

# $K$ -THEORY OF MODULI SPACES OF SHEAVES ON $\mathbb{P}^2$

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ABSTRACT. In this paper, we reinterpret the action of the shuffle algebra on the  $K$ -theory of moduli spaces of sheaves on  $\mathbb{P}^2$ . This will allow us to define  $K$ -theoretic analogues of Baranovsky's operators, via our so-called fine correspondences.

## 1. INTRODUCTION

Due to a classic result of Grojnowski and Nakajima ([4] and [5], for  $r = 1$ ) and Baranovsky ([1], for general  $r$ ), there is an action of the Heisenberg Lie algebra  $\widehat{\mathfrak{gl}}_1$  on the equivariant cohomology group  $H$  of the moduli space  $\mathcal{M} = \mathcal{M}(r)$  of rank  $r$  sheaves on a surface. This action is defined by constructing certain geometric correspondences  $a_n^\pm$ ,  $n \geq 1$  on  $\mathcal{M} \times \mathcal{M}$ .

Independently, Feigin-Tsybaliuk ([3]) and Schiffmann-Vasserot ([7]) introduced an action of a certain larger algebra  $\mathcal{A} \supset \widehat{\mathfrak{gl}}_1$  on the equivariant  $K$ -theory group of the space  $\mathcal{M}$ . This algebra is known by many names:

- the double cover of the shuffle algebra,
- the Hall algebra of an elliptic curve,
- the doubly-deformed  $W_{1+\infty}$ -algebra,
- the stabilizaton of the spherical part of Cherednik's DAHA,
- $U_q(\widehat{\mathfrak{gl}}_1)$ .

Each of the above names gives a different presentation of the algebra  $\mathcal{A}$ , and the relations between them are rather non-trivial. For example, in the present paper we will use the results of [6], which connects the first two presentations.

The action of  $\mathcal{A}$  on  $K$  constructed by [3] and [7] is defined on the generators of  $\mathcal{A}$  by certain simple correspondences. In our paper, we will use the more general **fine correspondences** in order to give explicit formulas for a much wider class of elements of  $\mathcal{A}$  acting on  $K$ . In particular, our main **Theorem 3.10** will provide a geometric description of  $\widehat{\mathfrak{gl}}_1 \subset \mathcal{A}$  acting on  $K$ , by explicitly constructing  $K$ -theoretic analogues of Baranovsky's operators  $a_n^\pm$ . Let us say a few things

about the structure of this paper:

- In Section 2 we introduce the moduli spaces of sheaves, their equivariant  $K$ -theory and tautological classes.
- In Section 3, we introduce the double shuffle algebra  $\mathcal{A}$  and the fine correspondences  $\mathfrak{Z}^k$ , in order to state our main result Theorem 3.10.
- In Section 4, we describe the geometry of the fine correspondences in more detail, and prove that they are lci.
- In Section 5, we will work out some fixed point computations, which will be used in Section 6 in order to compute the action of the fine correspondences on  $K$ . The goal is Theorem 6.10, which generalizes Theorem 3.10: **the fine correspondences act on  $K$  via certain explicit elements of the shuffle algebra.**
- In Section 7, we perform a similar computation to identify the geometric actions of the vector bundle  $E$  and the Lagrangian correspondence  $\mathfrak{V}^k$  (see Subsection 4.10) with the shuffle elements  $\varepsilon_k$  of [2].
- In Section 8, we show that the action of  $\mathcal{A}$  on  $K$  is well-defined. This has already been done in [3] and [7], but for the sake of completion, we will redo it in our language. We will also compute formulas for the matrix coefficients of shuffle elements in the fixed point basis.

I would foremost like to thank my advisor Andrei Okounkov for his help and advice during this project. In particular, the crucial definition of the fine correspondences of Section 4 was the product of many discussions with him on this subject. I would also like to thank Boris Feigin, Eugene Gorsky, Eric Vasserot and Alexander Tsymbaliuk for patiently explaining their viewpoints of this subject.

## 2. THE MODULI SPACE OF SHEAVES

**2.1.** Take the projective plane, and fix a line  $\infty \subset \mathbb{P}^2$ . Fix  $r \in \mathbb{N}$ , and let  $\mathcal{M}$  denote the moduli space of rank  $r$  torsion free sheaves  $\mathcal{F}$  on  $\mathbb{P}^2$ , together with an isomorphism (framing):

$$\mathcal{F}|_{\infty} \cong \mathcal{O}_{\infty}^{\oplus r}.$$

This latter condition forces  $c_1(\mathcal{F}) = 0$ , but  $c_2(\mathcal{F})$  is still free to range over the non-positive integers. For  $d \geq 0$ , we denote by  $\mathcal{M}_d \subset \mathcal{M}$  the connected component of sheaves of second Chern class  $-d$ . Its tangent space is:

$$T_{\mathcal{F}}\mathcal{M}_d = \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}(-\infty))$$

by the Kodaira-Spencer isomorphism. From this one can easily prove that  $\mathcal{M}_d$  is smooth of dimension  $2rd$ .

**2.2.** Let us fix coordinates  $(z_1, z_2)$  on  $\mathbb{P}^2$ , in which  $\infty$  is the line at infinity. The maximal torus  $H \subset GL_n$  and the product  $\mathbb{C}^* \times \mathbb{C}^*$  both act on  $\mathcal{M}_d$  by changing the trivialization at  $\infty$ , respectively by multiplying the base  $\mathbb{P}^2$  in the  $(z_1, z_2)$  coordinates. We will write  $T = H \times \mathbb{C}^* \times \mathbb{C}^*$ , and in this paper we will be mostly concerned with the  $T$ -equivariant  $K$ -theory rings:

$$K_T^*(\mathcal{M}_d),$$

which are all modules over  $\mathbb{K}_0 := \text{Sym}(T) = \mathbb{C}[x_1, \dots, x_n, q_1, q_2]$ . Throughout this paper, the field of constants will be:

$$\mathbb{K} := \text{Frac}(\mathbb{K}_0) = \mathbb{C}(x_1, \dots, x_n, q_1, q_2)$$

and we will study the  $\mathbb{K}$ -vector space:

$$K = \bigoplus_{d \geq 0} K_T^*(\mathcal{M}_d) \bigotimes_{\mathbb{K}_0} \mathbb{K}$$

**2.3.** One has a universal sheaf on  $\mathcal{M} \times \mathbb{P}^2$ . Pushing it forward along the standard projection gives us the **tautological sheaf**  $\mathcal{T}$  on  $\mathcal{M}$ :

$$\mathcal{T}|_{\mathcal{F}} = H^1(\mathbb{P}^2, \mathcal{F}(-\infty)) \quad (2.1)$$

which has rank  $d$  over the connected component  $\mathcal{M}_d$ . We will also be concerned with the virtual sheaf  $\mathcal{W}$ , defined by:

$$\mathcal{W} = \sum_{i=1}^r x_i^{-1} - (q_1 - 1)(q_2 - 1)\mathcal{T}. \quad (2.2)$$

We will only be interested in  $K$ -theory, so the above definition of  $\mathcal{W}$  is sufficient for our purposes. When we will need to emphasize the fact that  $\mathcal{T}$  and  $\mathcal{W}$  are sheaves over the connected component  $\mathcal{M}_d \subset \mathcal{M}$ , we will write  $\mathcal{T}_d$  and  $\mathcal{W}_d$  for them.

**2.4.** Take any class  $c \in K_T^*$  of any variety, and assume we can write it as:

$$c = \sum_i l_i - \sum_j l_j,$$

where  $l_i, l_j \in K_T^1$ . Then we define the rational function:

$$\Lambda(c, u) = \frac{\prod_i \left(1 - \frac{u}{l_i}\right)}{\prod_j \left(1 - \frac{u}{l_j}\right)} \quad (2.3)$$

We can apply this construction to the sheaves (2.1) and (2.2), and obtain rational functions:

$$\Lambda(\mathcal{T}, u), \quad \Lambda(\pm \mathcal{W}, u) \in K(u)$$

We call the coefficients of  $\Lambda(\mathcal{T}, u)$ , as well as all products thereof, **tautological classes**. They are important because of the following result, to be proved in

subsection 5.4.

**Proposition 2.5.** *For any  $d$ , the vector space:*

$$K_d := K_T^*(\mathcal{M}_d) \bigotimes_{\mathbb{K}_0} \mathbb{K}$$

*is spanned by tautological classes.*

### 3. THE DOUBLE SHUFFLE ALGEBRA

**3.1.** For any value of the sign  $\pm$ , consider an infinite set of variables  $z_{\pm 1}, z_{\pm 2}, \dots$ , and take the  $\mathbb{K}$ –vector space:

$$\bigoplus_{k \geq 0} \text{Sym}_{\mathbb{K}}(z_{\pm 1}, \dots, z_{\pm k}), \quad (3.1)$$

bigraded by  $k$  and homogenous degree. We can endow it with a  $\mathbb{K}$ –algebra structure by the multiplication:

$$\begin{aligned} & P(z_{\pm 1}, \dots, z_{\pm k}) * Q(z_{\pm 1}, \dots, z_{\pm l}) = \\ & = \text{Sym} \left[ P(z_{\pm 1}, \dots, z_{\pm k}) Q(z_{\pm(k+1)}, \dots, z_{\pm(k+l)}) \prod_{i=1}^k \prod_{k+1=j}^{k+l} \omega \left( \left( \frac{z_{\pm i}}{z_{\pm j}} \right)^{\pm 1} \right) \right] \end{aligned} \quad (3.2)$$

where:

$$\omega(x) = \frac{(x-1)(x-q)}{(x-q_1)(x-q_2)} \quad , \quad q = q_1 q_2 \quad (3.3)$$

and  $\text{Sym}$  denotes the symmetrization operator:

$$\text{Sym}(P(z_1, \dots, z_k)) = \sum_{\sigma \in S(k)} P(z_{\sigma(1)}, \dots, z_{\sigma(k)})$$

**3.2.** The **shuffle algebra**  $\mathcal{A}^{\pm}$  is defined as the subspace of (3.1) consisting of rational functions of the form:

$$P(z_{\pm 1}, \dots, z_{\pm k}) = \frac{\prod_{i \neq j} (z_{\pm i} - z_{\pm j}) \cdot p(z_{\pm 1}, \dots, z_{\pm k})}{\prod_{i \neq j} (z_{\pm i} - q_1 \cdot z_{\pm j})(z_{\pm i} - q_2 \cdot z_{\pm j})} \quad (3.4)$$

where  $p$  is a symmetric Laurent polynomial that satisfies the **wheel conditions**:

$$p(z, q_1 \cdot z, q_1 q_2 \cdot z, \text{anything}) = p(z, q_2 \cdot z, q_1 q_2 \cdot z, \text{anything}) = 0 \quad (3.5)$$

This condition is vacuous for  $k \leq 2$ . It is shown in [6] that  $\mathcal{A}^{\pm}$  is a subalgebra, and Conjecture 2.6 therein claims that it is equal to the subalgebra of (3.1) generated by the first graded piece  $\text{Sym}_{\mathbb{K}}(z_{\pm 1})$ .

**3.3.** Consider symbols  $h_n$  and  $h_{-n}$  for  $n \geq 0$ , and let us organize them into generating series:

$$h^\pm(z) = \sum_{n \geq 0} h_{\pm n} \cdot z^{\mp n}$$

We will let  $\mathcal{A}^0$  denote the commutative algebra generated by these symbols. Our main actor is the **double shuffle algebra**:

$$\mathcal{A} = \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+$$

on which we impose relations (3.6) and (3.7) below. To define these, let us introduce the notations:

$$\Omega(x) = \frac{\omega(x^{-1})}{\omega(x)} = \frac{(x - q^{-1})(x - q_1)(x - q_2)}{(x - q)(x - q_1^{-1})(x - q_2^{-1})}, \quad \alpha = (q_1 - 1)(q_2 - 1)(q^{-1} - 1)$$

Then we require that for all  $P^\pm \in \mathcal{A}^\pm$ , we have:

$$P^\pm(z_{\pm 1}, \dots, z_{\pm k}) * h^{\pm'}(z) = h^{\pm'}(z) * \left[ P(z_{\pm 1}, \dots, z_{\pm k}) \prod_{i=1}^k \Omega\left(\frac{z_{\pm i}}{z}\right)^{\pm 1} \right] \quad (3.6)$$

and:

$$\begin{aligned} [P^+(z_1, \dots, z_k), P^-(z_{-1}, \dots, z_{-l})] &= \sum_{a=1}^{\min(k, l)} \alpha^a : \text{Res} : \left[ \prod_{1 \leq i \neq j \leq a} \omega^{-1}\left(\frac{w_i}{w_j}\right) \right. \\ &\quad \left. \frac{h(w_1) \dots h(w_a)}{w_1 \dots w_a} \cdot \frac{P^-(z_{-1}, \dots, z_{-(l-a)}, w_1, \dots, w_a)}{\prod_{i=1}^a \prod_{j=1}^{l-a} \omega\left(\frac{z_{-j}}{w_i}\right)} * \frac{P^+(z_1, \dots, z_{k-a}, w_1, \dots, w_a)}{\prod_{i=1}^a \prod_{j=1}^{k-a} \omega\left(\frac{w_i}{z_j}\right)} \right] \end{aligned} \quad (3.7)$$

where  $\text{Res}$  refers to the residue at 0 minus the residue at  $\infty$  in the variables  $w_1, \dots, w_a$ , computed in this order. Depending on whether we are taking the residue at 0 or  $\infty$ , we need to put the sign  $-$  or  $+$  above each  $h(w_i)$  to give us the correct expansion.

**3.4.** To complete the picture, we need to explain what is meant by the “normal ordered residue”  $: \text{Res} :$  of (3.7). According to Conjecture 2.6 of [6], the shuffle element  $P^-$  can be written as a linear combination of:

$$\text{Sym} \left[ z_{-1}^{\mu_1} \dots z_{-l}^{\mu_l} \prod_{1 \leq i < j \leq l} \omega\left(\frac{z_{-j}}{z_{-i}}\right) \right] \quad (3.8)$$

for  $\mu_1, \dots, \mu_l \in \mathbb{Z}$ .<sup>1</sup> In each term of formula (3.7), we need to set some of these variables equal to  $w_1, \dots, w_a$ . By definition, the meaning of the time-ordered residue  $: \text{Res} :$  in (3.7) is that we only take the summands of (3.8) which keep

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<sup>1</sup>We do not even need to assume the Conjecture, since all the shuffle elements we care about are already of this form

the variables  $w_1, \dots, w_a$  in order. The shuffle element  $P$  is not affected by this convention.

**3.5.** The following theorem was proved, in a different formulation, in [3] and [7]:

**Theorem 3.6.** *There is an action of  $\mathcal{A}$  on  $K$ .*

We will prove this theorem in the Appendix, in a different way from *loc. cit.* To define this action, recall by Proposition 2.5 that it is enough to say how  $\mathcal{A}$  acts on tautological classes. For an arbitrary set of variables  $S$ , the subalgebra  $\mathcal{A}^0$  acts by multiplication:

$$h^\pm(z) \cdot \prod_{s \in S} \Lambda(\mathcal{T}_d, s) = \frac{\Lambda(\mathcal{W}_d, zq)}{\Lambda(\mathcal{W}_d, z)} \cdot \prod_{s \in S} \Lambda(\mathcal{T}_d, s) \quad (3.9)$$

where depending on whether the sign is  $-$  or  $+$ , the right hand side is expanded around  $z = 0$  or  $z = \infty$ . In particular, we see that  $h_{-0} = 1$  and  $h_{+0} = q^r$ . The upper/lower parts  $\mathcal{A}^\pm$  act by:

$$\begin{aligned} P(z_{\pm 1}, \dots, z_{\pm k}) \cdot \prod_{s \in S} \Lambda(\mathcal{T}_d, s) &= [\pm(q^{\pm 1} - 1)]^k \cdot \Lambda(\mathcal{T}_{d \pm k}, s) \\ &: \int : P(u_1, \dots, u_k) \prod_{i=1}^k \frac{\Lambda(\pm \mathcal{W}_{d \pm k}, u_i q^\varepsilon)}{\prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{\pm 1}} Du_1 \dots Du_k \end{aligned} \quad (3.10)$$

for any  $P \in \mathcal{A}^\pm$ , where  $Du = \frac{du}{2\pi i}$ . Here and throughout this paper,  $\varepsilon$  will denote 0 or 1, depending on whether the sign is  $-$  or  $+$ .

**3.7.** As before, we need to explain what is meant by the “normal ordered integral”  $: \int : .$  Because  $P$  is a shuffle element, the integrand of (3.10) has poles when  $u_i = q_1 u_j$  and  $u_i = q_2 u_j$  for all  $i \neq j$ . We **define** the time-ordered integral to integrate the variables  $u_1, \dots, u_k$  in this order, and only take into account the residues at the afore-mentioned poles for  $i < j$  (when the sign is  $+$ ) and for  $i > j$  (when the sign is  $-$ ). So we end up taking contributions from only half of the residues. More explicitly, according to Conjecture 2.6 of [6], we can write any shuffle element as a linear combination of:

$$P(z_{\pm 1}, \dots, z_{\pm k}) = \text{Sym} \left[ z_{\pm 1}^{\lambda_1} \dots z_{\pm k}^{\lambda_k} \prod_{1 \leq i < j \leq k} \omega \left( \left( \frac{z_{\pm i}}{z_{\pm j}} \right)^{\pm 1} \right) \right]$$

for  $\lambda_1, \dots, \lambda_k \in \mathbb{Z}$ . Then the normal ordered integral of (3.10) is **defined** to be:

$$\int u_1^{\lambda_1} \dots u_k^{\lambda_k} \prod_{1 \leq i < j \leq k} \omega \left( \left( \frac{u_i}{u_j} \right)^{\pm 1} \right) \prod_{i=1}^k \frac{\Lambda(\pm \mathcal{W}, u_i q^\varepsilon)}{\prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{\pm 1}} Du_1 \dots Du_k \quad (3.11)$$

where  $u_1, \dots, u_k$  are taken over contours which separate the sets  $\text{Poles}(\Lambda(\pm\mathcal{W}, u))$  and  $S \cup \{0, \infty\}$ . The contours must also be taken in order of the variables:  $u_1$  closest to the former set and  $u_k$  closest to the latter set.

**3.8.** In [2], the authors have introduced a commutative subalgebra  $\mathcal{C}^\pm \subset \mathcal{A}^\pm$ . This subalgebra consists of those shuffle elements (3.4) of total degree 0, with the property that:

$$\lim_{\xi \rightarrow \infty} P(\xi z_{\pm 1}, \dots, \xi z_{\pm i}, z_{\pm(i+1)}, \dots, z_{\pm k})$$

exists and is finite for all  $i$ . In [6], we show that:

$$\mathcal{C} = \mathcal{C}^- \otimes \mathcal{C}^+ \subset \mathcal{A}^- \otimes \mathcal{A}^+ = \mathcal{A}$$

is a Heisenberg algebra, meaning that  $\mathcal{C}^\pm = \mathbb{K}[P_{\pm 1}, P_{\pm 2}, \dots]$ , where:

$$[P_k, P_{-l}] = \delta_k^l \cdot (1 - (-q)^{rk})(q^{-k} - 1)(q_1^k - 1)(q_2^k - 1) \quad \forall k, l \geq 1$$

In fact,  $P_{\pm k}$  are uniquely determined (up to constant multiple) as shuffle elements of “minimal degree”, a notion which is explained in [6]. The action of  $\mathcal{C} \subset \mathcal{A}$  on  $K$  is simply the  $q$ -deformed version of the Nakajima-Grojnovsky-Baranovsky action.

**3.9.** In particular, [6] proves the following formulas for the Heisenberg generators:

$$P_k = c_k \cdot \text{Sym} \left[ \frac{1 + q^{\frac{z_k}{z_{k-1}}} + \dots + q^{k-1} \frac{z_k}{z_1}}{z_1^{-1} \left( \frac{z_1}{z_2} - q \right) \dots \left( \frac{z_{k-1}}{z_k} - q \right) z_k} \prod_{i < j} \omega \left( \frac{z_i}{z_j} \right) \right] \quad (3.12)$$

$$P_{-k} = c_k \cdot \text{Sym} \left[ \frac{1 + q^{\frac{z_{-(k-1)}}{z_{-k}}} + \dots + q^{k-1} \frac{z_{-1}}{z_{-k}}}{z_{-1} \left( \frac{z_{-2}}{z_{-1}} - q \right) \dots \left( \frac{z_{-k}}{z_{-(k-1)}} - q \right) z_{-k}^{-1}} \prod_{i < j} \omega \left( \frac{z_{-j}}{z_{-i}} \right) \right] \quad (3.13)$$

where:

$$c_k = \frac{-\alpha}{(1 - q^{-1})^k (q_1^k - 1)(q_2^k - 1)}$$

By Theorem 3.6, these operators act on  $K$ . Our main task in this paper is to give a geometric description of this action. We consider the **fine correspondence**:

$$\mathfrak{Z}_d^k = \{(\mathcal{F}^0 \supset_x \mathcal{F}^1 \supset_x \dots \supset_x \mathcal{F}^k), \quad x \in D\} \subset \mathcal{M}_d \times \mathcal{M}_{d+1} \times \dots \times \mathcal{M}_{d+k}$$

where  $\supset_x$  means that the quotient of the two sheaves must be supported at the point  $x$ , and  $D$  is the line  $z_2 = 0$  in the complex plane. This variety comes with two projection maps  $p^-, p^+ : \mathfrak{Z}_d^k \rightarrow \mathcal{M}_d, \mathcal{M}_{d+k}$ . It will be shown in Subsection 4.9 that  $\mathfrak{Z}_d^k$  is lci<sup>2</sup>, and in Subsection 4.2 we will define the tautological line bundles  $\mathcal{L}_1, \dots, \mathcal{L}_k$  on  $\mathfrak{Z}_d^k$ . Let us write  $l_1, \dots, l_k \in K_T^1(\mathfrak{Z}_d^k)$  for the  $K$ -theory classes

<sup>2</sup>More precisely, we will show that its  $K$ -theoretic class is a linear combination of lci varieties, which makes it suitable for  $K$ -theoretic computations

of these line bundles. Then our main result reads as follows:

**Theorem 3.10.** *The Heisenberg operators  $P_{\pm k}$  act on  $K$  via the correspondences:*

$$a_{\pm k}(\gamma) = c_k^{\pm} \cdot p_*^{\pm} \left[ (l_1 \dots l_k)^{-r\varepsilon} \left( 1 + q \frac{l_k^{\pm 1}}{l_{k-1}^{\pm 1}} + \dots + q^{k-1} \frac{l_k^{\pm 1}}{l_1^{\pm 1}} \right) p^{\mp*}(\gamma) \right]$$

for all  $\gamma \in K$ , and the constants  $c_k^{\pm}$  are given by:

$$c_k^+ = \frac{\alpha \cdot [(-q)^r x_1 \dots x_r]^k (1 - q_1^{-1})}{(1 - q_1^{-k})(1 - q_2^{-k})}, \quad c_k^- = \frac{\alpha \cdot (1 - q_1)}{(q_1^k - 1)(q^k - q_1^k)}$$

Recall that  $\varepsilon$  is 0 or 1, depending on whether the sign is  $-$  or  $+$ .

**3.11.** The elliptic Hall algebra, which was hinted about in the introduction, naturally carries an action of  $SL_2(\mathbb{Z})$  by automorphisms. This algebra is isomorphic to the double shuffle algebra, and this isomorphism is made rather explicit in [6]. There, we obtain explicit formulas for the shuffle elements:

$$P_{k,d} = \gamma \cdot P_n,$$

where  $\gamma$  is any element of  $SL_2(\mathbb{Z})$  that takes the vector  $(n, 0)$  to  $(k, d)$ . Eugene Gorsky has explained to us that the action of the  $P_{k,d}$  on  $K$  (known in physics as the Fock space) is important in knot theory (and its physical relative, Chern-Simons theory). Therefore, in section 8.5, we will give explicit formulas for this action.

**3.12.** At the suggestion of Boris Feigin, we will prove the following result:

**Proposition 3.13.** *The vector  $v = \sum_{d \geq 0} 1_d \in K$  is an eigenvector of the Heisenberg annihilators:*

$$a_{-k} \cdot v = \frac{(-1)^k \alpha}{(q_1^k - 1)(q_2^k - 1)} \cdot v$$

**Proof** We can simply apply (3.10) and (3.13), which tell us that  $a_{-k} \cdot 1_d$  equals:

$$c_k(1 - q^{-1})^k \int \frac{u_k + qu_{k-1} + \dots + q^{k-1}u_1}{u_1 \left( \frac{u_2}{u_1} - q \right) \dots \left( \frac{u_k}{u_{k-1}} - q \right)} \prod_{i < j} \omega \left( \frac{u_j}{u_i} \right) \prod_{i=1}^k \Lambda(-\mathcal{W}_{d-k}, u_i) Du_1 \dots Du_k$$

where the contours go around the poles of the rational function  $\Lambda(-\mathcal{W})$ , with  $u_1$  being the innermost one. We can deform the contours to small loops around 0 and  $\infty$ , with  $u_k$  being the innermost contour. There is no residue at  $\infty$  in  $u_k$ , because  $\Lambda(-\mathcal{W}, u)$  vanishes to order  $r \geq 1$  there. As for the residue at 0 in  $u_k$ , it equals:

$$-c_k(1 - q^{-1})^k \int \frac{u_{k-1} + \dots + q^{k-2}u_1}{u_1 \left( \frac{u_2}{u_1} - q \right) \dots \left( \frac{u_{k-1}}{u_{k-2}} - q \right)} \prod_{i < j} \omega \left( \frac{u_j}{u_i} \right) \prod_{i=1}^{k-1} \Lambda(-\mathcal{W}_{d-k}, u_i) Du_1 \dots Du_{k-1}$$



The same argument applies here in  $u_{k-1}$ , so we can repeat it to obtain  $(-1)^{k-1}c_k(1-q^{-1})^k \cdot 1_{d-k}$ . This yields the desired result.  $\square$

**3.14.** In [2], the authors introduce the following elements of  $\mathcal{C}$ :

$$\varepsilon_k^{q^\pm} = \prod_{1 \leq i \neq j \leq k} \omega \left( \frac{z_{\pm j}}{z_{\pm i}} \right) \in \mathcal{C}^\pm \quad (3.14)$$

and:

$$\varepsilon_k^{q_x^\pm} = \prod_{1 \leq i \neq j \leq k} \frac{z_{\pm j} - z_{\pm i}}{z_{\pm j} - q_x^{-1} z_{\pm i}} \in \mathcal{C}^\pm, \quad \forall x \in \{1, 2\} \quad (3.15)$$

In section 4.10, we will construct geometric versions of these, i.e. geometric operators that act on  $K$  via the above elements of  $\mathcal{C} \subset \mathcal{A}$ .

#### 4. THE FINE CORRESPONDENCES

**4.1.** In cohomology, Baranovsky's operators  $a_k^\pm$  are defined using the correspondences:

$$\mathfrak{C}_d^k = \{(\mathcal{F} \supset_x \mathcal{F}'), \quad x \in D\} \subset \mathcal{M}_d \times \mathcal{M}_{d+k}$$

where  $D \subset \mathbb{P}^2$  is a fixed line, and  $\supset_x$  means that the quotient of the two sheaves is supported at the point  $x$ . The problem with using these correspondences in the present  $K$ -theoretic setting is that they are not smooth. Instead, we will consider the **fine correspondences**:

$$\mathfrak{Z}_d^k = \{(\mathcal{F}^0 \supset_x \mathcal{F}^1 \supset_x \dots \supset_x \mathcal{F}^k), \quad x \in D\} \subset \mathcal{M}_d \times \mathcal{M}_{d+1} \times \dots \times \mathcal{M}_{d+k}$$

We will often write  $\mathfrak{Z}^k = \sqcup_d \mathfrak{Z}_d^k$  if the degree will be of no concern, and the same for  $\mathfrak{C}^k$ . Then the map from  $\mathfrak{Z}^k$  to  $\mathfrak{C}^k$  that forgets the intermediate sheaves in the flag is birational and proper, and therefore  $\mathfrak{Z}^k$  can be used instead of  $\mathfrak{C}^k$  to define the Baranovsky operators in cohomology.

**4.2.** The variety  $\mathfrak{Z}^k$  is itself not smooth, but we will show in subsection 4.9 that its  $K$ -theoretic class is a linear combination of lci varieties. Therefore, we can treat  $\mathfrak{Z}^k$  as a linear combination of vector bundles on a smooth space for  $K$ -theoretic computations. Moreover,  $\mathfrak{Z}^k$  has much more geometric information than  $\mathfrak{C}^k$ , namely  $k$  tautological line bundles:

$$\mathcal{L}_i|_{(\mathcal{F}^0 \supset_x \mathcal{F}^1 \supset_x \dots \supset_x \mathcal{F}^k)} = \mathrm{R}\Gamma(\mathbb{P}^2, \mathcal{F}^i / \mathcal{F}^{i-1}). \quad (4.1)$$

for all  $i \in \{1, \dots, k\}$ . Let  $l_i = [\mathcal{L}_i] \in K_T^1(\mathfrak{Z}^k)$ , and also denote by  $p^-/p^+ : \mathfrak{Z}^k \rightarrow \mathcal{M}$  the two projections to the first/last sheaf in the flag. Then for any Laurent polynomial  $m(z_1, \dots, z_k)$ , we define the operator:

$$a_m^\pm : K \longrightarrow K, \quad \alpha \longrightarrow p_*^\pm (m(l_1, \dots, l_k) \cdot p^{\mp*}(\alpha)) \quad (4.2)$$

**4.3.** In the rest of this section, we will introduce some more geometry, much of which is not necessary for our proof of Theorem 3.6. We will prove the few results that we will actually use, but include more in order to paint a complete picture. Consider the variety:

$$\mathfrak{U}_d^k = \{(\mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^k \supset \mathcal{F}^0(-D))\} \subset \mathcal{M}_d \times \mathcal{M}_{d+1} \times \dots \times \mathcal{M}_{d+k}$$

<sup>3</sup> Points  $\overline{\mathcal{F}} \in \mathfrak{U}_d^k$  are flags of sheaves, but we may realize them in a different way by considering the  $k+1$  to 1 map:

$$\sigma : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \quad \sigma(z_1, z_2) = (z_1, z_2^{k+1}).$$

where coordinates are taken in such a way that  $D = \{z_2 = 0\}$ . Then to any flag of sheaves  $\mathcal{F}$  we can associate the following rank  $r$  torsion free sheaf in  $\mathcal{M}_{\tilde{d}}$ , where  $\tilde{d} = d + (d+1) + \dots + (d+k)$ :

$$\tilde{\mathcal{F}} = \sigma^* \mathcal{F}_k + \sigma^* \mathcal{F}_{k-1}(-D) + \dots + \sigma^* \mathcal{F}_0(-kD). \quad (4.3)$$

The sum is **not** a direct sum, but it refers to the convex hull as subsheaves of  $\sigma^* \mathcal{F}_0$ . The finite group  $G = \mathbb{Z}/(k+1)\mathbb{Z}$  acts on  $\mathcal{M}_{\tilde{d}}$  by multiplication on the second coordinate, and the sheaf  $\tilde{\mathcal{F}}$  will be invariant under this action. We therefore obtain a well-defined map:

$$\mathfrak{U}_d^k \longrightarrow \mathcal{M}_{\tilde{d}}^G, \quad \overline{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}}. \quad (4.4)$$

**4.4.** We can construct an inverse to the map (4.4): take a  $G$ -invariant sheaf  $\tilde{\mathcal{F}} \in \mathcal{M}_{\tilde{d}}$  and define  $\mathcal{F}_l$  to be its  $\exp(-2\pi il/(k+1))$  isotypic component. Then the sheaves  $\mathcal{F}_0, \dots, \mathcal{F}_k$  thus constructed paste together into a flag  $\overline{\mathcal{F}} \in \mathfrak{U}_d^k$ , and this is the inverse of the map (4.4). Therefore, each  $\mathfrak{U}_d^k$  is isomorphic to one of the connected components of  $\mathcal{M}_{\tilde{d}}^G$ . This allows us to compute the tangent space to any  $\overline{\mathcal{F}} \in \mathfrak{U}_d^k$ : it is merely the  $G$ -invariant part of the tangent space to  $\mathcal{M}_{\tilde{d}}$  at the corresponding point, in other words:

$$T_{\overline{\mathcal{F}}} \mathfrak{U}_d^k = \text{Ext}^1(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}(-\infty))^G \quad (4.5)$$

It is easy to see that the corresponding Hom and  $\text{Ext}^2$  vanish, and this has the very important implication that  $\mathfrak{U}_d^k$  is **smooth**. Its dimension is  $2rd + rk$ , as we will see from the tangent space computation of Subsection 5.7.

---

<sup>3</sup>The contents of this section hold for any assumption on the degrees in the above expression, though we will only need degrees  $(d, d+1, \dots, d+k)$

**4.5.** We can actually construct more complicated varieties. For any ordered partition  $\mu = (\mu_1, \dots, \mu_t)$  of  $k$  with  $\mu_i \geq 1$ , we define:

$$\mathfrak{Z}_d^\mu = \overline{\{(\mathcal{F}^1 \supset \mathcal{F}^2 \supset \dots \supset \mathcal{F}^{k+1})\}} \subset \mathcal{M}_d \times \mathcal{M}_{d+1} \times \dots \times \mathcal{M}_{d+k}$$

where the  $k$  support points in the above sheaf inclusions are all required to be on  $D$ , and they have to coincide in successive chunks of  $\mu_1, \mu_2, \dots, \mu_t$ , but must differ among the chunks. Note that:

$$\mathfrak{Z}_d^{(k)} = \mathfrak{Z}_d^k, \quad \mathfrak{Z}_d^{(1, \dots, 1)} = \mathfrak{U}_d^k$$

All the  $\mathfrak{Z}_d^\mu$  are irreducible of dimension  $2rd + rk$ , and now we will see how their  $K$ -theoretic classes are linear combinations of lci subvarieties of  $\mathcal{M}_d \times \dots \times \mathcal{M}_{d+k}$ .

**4.6.** Consider the following sheaf on  $\mathcal{M}_d \times \mathcal{M}_{d'}$ :

$$E|_{\mathcal{F}, \mathcal{F}'} = \text{Ext}^1(\mathcal{F}', \mathcal{F}(-\infty)) \quad (4.6)$$

The corresponding Hom vanishes because of the twist by  $\infty$ , and the corresponding  $\text{Ext}^2$  vanishes by the same reason and Serre duality. Therefore,  $E$  is a vector bundle, and in Subsection 5.5 we will see that its rank is  $r(d+d')$ .

**Proposition 4.7.** *The vector bundle  $E$  has a section  $s$  which vanishes set-theoretically on the locus:*

$$\mathfrak{C} = \{(\mathcal{F} \supset \mathcal{F}')\} \subset \mathcal{M}_d \times \mathcal{M}_{d'}$$

**Proof** Consider the short exact sequence:

$$0 \longrightarrow \mathcal{F}(-\infty) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}|_\infty \longrightarrow 0$$

and the long exact Hom sequences it induces:

$$\dots \longrightarrow \text{Hom}(\mathcal{F}', \mathcal{F}) \longrightarrow \text{Hom}(\mathcal{F}', \mathcal{F}|_\infty) \longrightarrow \text{Ext}^1(\mathcal{F}', \mathcal{F}(-\infty)) \longrightarrow \dots$$

Because our sheaves come with a tautological trivialization at  $\infty$ , we have a natural identification  $\mathcal{F}|_\infty \cong \mathcal{F}'|_\infty$  for any pair of sheaves  $(\mathcal{F}, \mathcal{F}')$ . This means that we have a canonically defined element in the middle Hom space, and we define our section  $s \in \text{Ext}^1(\mathcal{F}', \mathcal{F}(-\infty))$  by pushing this element forward under the above exact sequence.

By the exact sequence, the section vanishes precisely when the canonical morphism  $\mathcal{F}' \longrightarrow \mathcal{F}'|_\infty \cong \mathcal{F}|_\infty$  comes from a morphism  $\mathcal{F}' \longrightarrow \mathcal{F}$ . Because the sheaves have the same first Chern class, this can only happen when  $\mathcal{F}' \subset \mathcal{F}$ .  $\square$

*Remark 4.8.* Note that when  $d = d'$ , the locus is simply the diagonal and thus has the expected dimension. We conclude that  $e(E) = c \cdot [\Delta]$  scheme-theoretically, for some integer  $c \geq 1$ . Though we will not need this, we claim that this integer is 1.

**4.9.** Now we are finally able to explain why the subvarieties  $\mathfrak{Z}_d^\mu$  of  $\mathcal{M}_d \times \dots \times \mathcal{M}_{d+k}$  are linear combinations of lci subvarieties. Every ordered partition  $\mu$  of  $k$  can be perceived as a  $k \times 1$  rectangle where we draw some black vertical bars to separate the parts of the partition  $\mu$ . The remaining vertical bars (which we will call white) determine another ordered partition  $\lambda$  of  $k$ , simply by inverting black and white:



In the above example,  $k = 11$  and:

$$\mu = (3, 1, 1, 2, 4) \longrightarrow \lambda = (1, 1, 4, 2, 1, 1)$$

With this in mind, we consider the smooth variety:

$$\mathfrak{U}_d^{\lambda_1} \times \mathfrak{U}_{d+\lambda_1}^{\lambda_2} \times \dots \times \mathfrak{U}_{d+\lambda_1+\dots+\lambda_{t-1}}^{\lambda_t} \quad (4.7)$$

of dimension  $2r[d + (d + \lambda_1) + \dots + (d + \lambda_1 + \lambda_2 + \dots + \lambda_{t-1})] + rk$ , and the following vector bundle on it:

$$\tilde{E} = \bigoplus_{i=1}^{t-1} \text{Ext}^1(\mathcal{F}_i^+, \mathcal{F}_{i+1}^-)$$

where  $\mathcal{F}_i^- / \mathcal{F}_i^+$  denote the first / last sheaf in  $\mathfrak{U}_{d+\lambda_1+\dots+\lambda_{i-1}}^{\lambda_i}$ . The rank of this vector bundle is equal to  $2r[(d + \lambda_1) + \dots + (d + \lambda_1 + \dots + \lambda_{t-1})]$ , and we can consider the direct sum of the sections  $s$  constructed in Proposition 4.7. By Remark 4.8, this direct sum vanishes scheme-theoretically <sup>4</sup> on the locus:

$$\{(\mathcal{F}_1 \supset \dots \supset \mathcal{F}_{k+1})\} \subset \mathfrak{U}_d^{\lambda_1} \times \mathfrak{U}_{d+\lambda_1}^{\lambda_2} \times \dots \times \mathfrak{U}_{d+\lambda_1+\dots+\lambda_{t-1}}^{\lambda_t}$$

This locus is the union of  $\mathfrak{Z}_d^{\mu'}$  for all partitions  $\mu' \leq \mu$ . The notation  $\mu' \leq \mu$  means that the partition  $\mu'$  is obtained from  $\mu$  by breaking up some of its parts into smaller ones. All these irreducible components have dimension  $2rd + rk$ , precisely the expected dimension of the locus. We conclude that we have the following equality in the Chow group:

$$[\mathfrak{Z}_d^\mu] + \sum_{\mu' < \mu} [\mathfrak{Z}_d^{\mu'}] = e(\tilde{E})$$

and therefore the left hand side is lci inside (4.7). Since the space (4.7) is smooth, we conclude that the left hand side is lci inside  $\mathcal{M}_d \times \dots \times \mathcal{M}_{d+k}$ . Starting with  $\mathfrak{Z}_d^{(1,\dots,1)} = \mathfrak{U}_d^k$  which is smooth to begin with, this inductively expresses each  $\mathfrak{Z}_d^\mu$  as a linear combination of lci classes inside  $\mathcal{M}_d \times \dots \times \mathcal{M}_{d+k}$ . Ultimately, we get such an expression for  $\mathfrak{Z}_d = \mathfrak{Z}_d^{(k)}$ , and this allows us to properly define the operators (4.2).

<sup>4</sup>Technically, we have only proved **scheme**-theoretic vanishing up to a positive integer  $c$ , but this will be enough for our needs

**4.10.** The vector bundle  $E$  of (4.6) has many more uses than just as a technical tool. Consider the projections  $\bar{p}^-$  and  $\bar{p}^+$  from  $\mathcal{M}_d \times \mathcal{M}_{d+k}$  to  $\mathcal{M}_d$  and  $\mathcal{M}_{d+k}$ , respectively, and consider the operators:

$$b_k^\pm : K \longrightarrow K, \quad b_k^\pm(\alpha) = \bar{p}_*^\pm(\Lambda(E, 1) \cdot \bar{p}^{*\mp}(\alpha))$$

where  $\Lambda$  is the construction of subsection (2.4). We can also consider the locus:

$$\mathfrak{V}_d^k = \{(\mathcal{F} \supset \mathcal{F}' \supset \mathcal{F}(-D))\} \subset \mathcal{M}_d \times \mathcal{M}_{d+k} \quad (4.8)$$

which is smooth by the same argument as in subsection 4.4. As pointed out by Andrei Okounkov, this locus is Lagrangian. We have projections  $\tilde{p}^-$  and  $\tilde{p}^+$  from  $\mathfrak{V}_d^k$  to  $\mathcal{M}_d$  and  $\mathcal{M}_{d+k}$ , which allow us to define operators:

$$c_k^\pm : K \longrightarrow K, \quad c_k^\pm(\alpha) = \tilde{p}_*^\pm(\tilde{p}^{*\mp}(\alpha))$$

**Proposition 4.11.** *Under the action of Theorem 3.6, the geometric operators  $b_k^\pm$  and  $c_k^\pm$  and the algebraic operators  $\varepsilon_k^{q^\pm}$  and  $\varepsilon_k^{q_2^\pm}$  are connected by:*

$$b_k^\pm = \left[ \frac{-\rho_\pm}{(q_1^{\pm 1} - 1)(q_2^{\pm 1} - 1)} \right]^k \varepsilon_k^{q^\pm}(z_{\pm 1}, \dots, z_{\pm k}) \cdot (z_{\pm 1} \dots z_{\pm k})^{-r\varepsilon} \in \mathcal{A}^\pm \quad (4.9)$$

$$c_k^\pm = \left[ \frac{\rho_\pm}{(q_1^{\pm 1} - 1)(q^{\pm 1} - 1)} \right]^k \varepsilon_k^{q_2^\pm}(z_{\pm 1}, \dots, z_{\pm k}) \cdot (z_{\pm 1} \dots z_{\pm k})^{-r\varepsilon} \in \mathcal{A}^\pm \quad (4.10)$$

If we replace the horizontal divisor  $D = \{z_2 = 0\}$  by the vertical divisor  $\{z_1 = 0\}$  in the definition of  $c_k^\pm$ , then we should replace  $q_2$  with  $q_1$  in the above.

## 5. FIXED POINTS

**5.1.** This section consists of fixed point computations, which can be skipped on a first reading, if the reader is willing to accept the results herein in the next section. We will talk about the  $T$  fixed points of  $\mathcal{M}_d$ . These are indexed by  $r$ -tuples of partitions  $\lambda = (\lambda^1, \dots, \lambda^r)$  whose sizes sum up  $d$ , and are given by:

$$\mathcal{I}_\lambda := \mathcal{I}_{\lambda^1} \oplus \dots \oplus \mathcal{I}_{\lambda^r}$$

with the direct sum diagonal in our fixed maximal torus  $H \subset GL_r$ . In the above, for any partition  $\lambda^i = (\lambda_0^i \geq \lambda_1^i \geq \dots)$ , we take the ideal:

$$\mathcal{I}_{\lambda^i} = (z_1^{\lambda_0^i} z_2^0, z_1^{\lambda_1^i} z_2^1, \dots) \subset \mathbb{C}[z_1, z_2]$$

Fixed points are important because equivariant  $K$ -theory is “concentrated” around these fixed points, in the sense that any relation between  $K$ -theory classes holds if and only if it holds when restricted to all fixed points.

**5.2.** Since we are using the language of partitions, we will often use their associated Young diagrams. For example, Figure 5.3 depicts the Young diagram of the partition  $(4, 3, 1)$ . It consists of  $8 = 4 + 3 + 1$  boxes, which we will often denote by  $\square$ . The content  $\chi(\square)$  of a box is defined to be  $q_1^i q_2^j$ , where  $(i, j)$  are the integer coordinates of the lower left corner of the box.

The vertical/horizontal leg length of a box  $\square$  relative to the partition  $\lambda$  is denoted by  $v_\lambda(\square)/h_\lambda(\square)$ , and is defined to be the number of boxes in  $\lambda$  that are above/to the right of  $\square$  including itself. For example, for the box marked by  $X$  in Figure 5.3, we have  $v_\lambda(\square) = 2$  and  $h_\lambda(\square) = 3$ . The notion of leg length can be defined for boxes outside the partition as well, but in that case the corresponding number is non-positive. For example, for a box placed in an inner corner of a partition (e.g. the box whose lower left corner is  $(1, 2)$  in Figure 5.3), both the vertical and horizontal leg lengths are 0.

**5.3.** We will now need to compute the characters of  $T$  acting in various vector spaces corresponding to fixed points  $\lambda$ . The first of these is the character in the fibers of the tautological bundle:

$$\text{char}_T(\mathcal{T}|\lambda) = \sum_{k=1}^r x_k^{-1} \sum_{\square \in \lambda^k} \chi(\square)$$

Therefore, the character in the fiber of  $\mathcal{W}$  is:

$$\begin{aligned} \text{char}_T(\mathcal{W}|\lambda) &= \sum_{k=1}^r x_k^{-1} \left[ 1 - (q_1 - 1)(q_2 - 1) \sum_{\square \in \lambda^k} \chi(\square) \right] = \\ &= \sum_{k=1}^r x_k^{-1} \sum_{\substack{\square \text{ inner} \\ \text{corner of } \lambda^k}} \chi(\square) - \sum_{k=1}^r x_k^{-1} \sum_{\substack{\square \text{ outer} \\ \text{corner of } \lambda^k}} \chi(\square) \end{aligned} \quad (5.1)$$

where the notion of outer and inner corner of a partition is exemplified in the following diagram, for  $\lambda = (4, 3, 1)$ . The outer corners are the boxes whose lower left corner is a solid circle, while the inner corners correspond to a hollow circle:

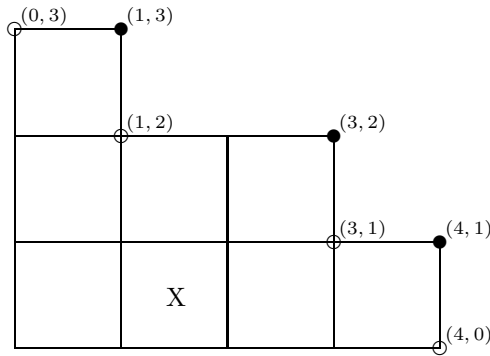


Figure 5.3

**5.4.** We will now use this description to prove Proposition 2.5. It is enough to find a single  $K$ -theory class such that anything in  $K_T^*(\mathcal{M}_d)$  is a linear combination of the powers of this class. Since the Vandermonde determinant is non-zero, it is enough for this class to have different restrictions to all fixed points  $\lambda$  of  $\mathcal{M}_d$ . It is easy to see that such a class is precisely:

$$\Lambda_1(\mathcal{T}) = \text{next to highest term of } \Lambda(\mathcal{T}, u),$$

$$\Lambda^1(\mathcal{T})|_\lambda = \pm \text{char}_T(\mathcal{T}^\vee|_\lambda) = \pm \sum_{k=1}^r x_k \sum_{\square \in \lambda^k} \chi(\square)^{-1}$$

**5.5.** The character in the fiber of the vector bundle  $E$  at a fixed point  $(\mathcal{I}_\lambda, \mathcal{I}_\mu)$  equals:

$$\begin{aligned} \text{char}_T(E_{\lambda, \mu}) &= \sum_{i=1}^r \sum_{i'=1}^r \frac{x_i}{x_{i'}} \sum_{j=0}^{\infty} \sum_{j'=0}^{\infty} q_1 \frac{(q_1^{\mu_{j'}^{i'}} - 1)(q_1^{-\lambda_j^i} - 1)}{q_1 - 1} q_2^{j'-j} (q_2 - 1) + \\ &+ \sum_{i=1}^r \sum_{i'=1}^r \frac{x_i}{x_{i'}} \sum_{j=0}^{\infty} \frac{q_1^{-\lambda_j^i} - 1}{q_1^{-1} - 1} q_2^{-j} + \sum_{i=1}^r \sum_{i'=1}^r \frac{x_i}{x_{i'}} \sum_{j'=0}^{\infty} q_1 \frac{q_1^{\mu_{j'}^{i'}} - 1}{q_1 - 1} q_2^{j'+1} \end{aligned} \quad (5.2)$$

De-localizing, i.e. using an equality at the level of fixed points to imply an equality of global  $K$ -theory classes, gives us:

$$\begin{aligned} [E] &= [\mathcal{T}_1]^\vee \sum_{i=1}^r x_i^{-1} + [\mathcal{T}_2] \sum_{i=1}^r q x_i - (q_1 - 1)(q_2 - 1) \cdot [\mathcal{T}_1]^\vee \otimes [\mathcal{T}_2] = \\ &= [\mathcal{T}_1]^\vee \otimes [\mathcal{W}_2] + \sum_{i=1}^r q x_i \cdot [\mathcal{T}_2] = q[\mathcal{W}_1]^\vee \otimes [\mathcal{T}_2] + [\mathcal{T}_1]^\vee \sum_{i=1}^r x_i^{-1} \end{aligned} \quad (5.3)$$

where  $\mathcal{T}_i = p_i^*(\mathcal{T})$  and  $\mathcal{W}_i = p_i^*(\mathcal{W})$  for  $i \in \{1, 2\}$ , and  $p_1, p_2$  are the projections onto the first and second factors. An equivalent formula without minus signs is:

$$\text{char}_T(E_{\lambda, \mu}) = \sum_{i=1}^r \sum_{j=1}^r \frac{x_i}{x_j} \left( \sum_{\square \in \mu^j} q_1^{-h_{\lambda^i}(\square)+1} q_2^{v_{\mu^j}(\square)} + \sum_{\square \in \lambda^i} q_1^{h_{\mu^j}(\square)} q_2^{-v_{\lambda^i}(\square)+1} \right) \quad (5.4)$$

In particular,  $\text{char}_T(E_{\lambda, \mu})$  will contain the trivial character 1 **unless**  $\lambda \leq \mu$ .<sup>5</sup> This remark will be very useful in section 7.

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<sup>5</sup>Here,  $\lambda \leq \mu$  means that  $\lambda_i \leq \mu_i$  for all  $i$ , or equivalently,  $\mathcal{I}_\mu \subset \mathcal{I}_\lambda$

**5.6.** Setting  $\lambda = \mu$ , the above gives us the character of  $T$  in the tangent space of  $\mathcal{M}$ :

$$\begin{aligned} [T\mathcal{M}] &= [\mathcal{T}]^\vee \otimes [\mathcal{W}] + \sum_{i=1}^r qx_i \cdot [\mathcal{T}] = \\ &= [\mathcal{T}]^\vee \sum_{i=1}^r x_i^{-1} + [\mathcal{T}] \sum_{i=1}^r qx_i - (q_1 - 1)(q_2 - 1) \cdot [\mathcal{T}]^\vee \otimes [\mathcal{T}] \end{aligned} \quad (5.5)$$

For the restriction to the fixed point  $\lambda$ , (5.4) gives us:

$$\text{char}_T(T_\lambda \mathcal{M}) = \sum_{i=1}^r \sum_{j=1}^r \frac{x_i}{x_j} \left( \sum_{\square \in \lambda^j} q_1^{-h_{\lambda^i}(\square)+1} q_2^{v_{\lambda^j}(\square)} + \sum_{\square \in \lambda^i} q_1^{h_{\lambda^j}(\square)} q_2^{-v_{\lambda^i}(\square)+1} \right) \quad (5.6)$$

and the symplectic form on  $T\mathcal{M}$  pairs the dual one-dimensional vector spaces:

$$\frac{x_i}{x_j} \cdot q_1^{-h_{\lambda^i}(\square)+1} q_2^{v_{\lambda^j}(\square)} \quad \text{and} \quad \frac{x_j}{x_i} \cdot q_1^{h_{\lambda^i}(\square)} q_2^{-v_{\lambda^j}(\square)+1}, \quad \forall \square \in \lambda^j$$

As such, we see that the symplectic form has equivariant weight  $q_1 q_2 = q$ .

**5.7.** Let us now look at the varieties  $\mathcal{U}_d^k$  of Subsection 4.3, whose torus fixed points are flags of torsion-free sheaves:

$$\mathcal{F} \quad : \quad \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots \supset \mathcal{F}^k \supset \mathcal{F}^0(-D) \quad (5.7)$$

As in subsection 4.3, the tangent space to this variety is the  $\mathbb{Z}/(k+1)\mathbb{Z}$  invariant part of the tangent space to  $\mathcal{M}$  at the point  $\tilde{\mathcal{F}}$  of (4.3). By (5.5), this tangent space equals:

$$[\tilde{\mathcal{T}}]^\vee \sum_{i=1}^r \frac{1}{x_i} + [\tilde{\mathcal{T}}] \sum_{i=1}^r qx_i - (q_1 - 1)(q_2 - 1) \cdot [\tilde{\mathcal{T}}]^\vee \otimes [\tilde{\mathcal{T}}] \quad (5.8)$$

where  $[\tilde{\mathcal{T}}] = [\mathcal{T}_0] \oplus \dots \oplus [\mathcal{T}_k]$ , and  $\mathcal{T}_l$  denotes the tautological vector bundle on the moduli space corresponding to  $\mathcal{F}^l$ . The generator  $1 \in \mathbb{Z}/(k+1)\mathbb{Z}$  multiplies the summand  $[\mathcal{T}_l]$  by the root of unity  $\zeta^l$  and multiplies  $q_2$  by  $\zeta$ . Therefore, the  $\mathbb{Z}/(k+1)\mathbb{Z}$  invariant part of (5.8) equals:

$$[T_{\mathcal{F}} \mathcal{U}_d^k] = [\mathcal{T}_0]^\vee \sum_{i=1}^r \frac{1}{x_i} + [\mathcal{T}_k] \sum_{i=1}^r qx_i - (q_1 - 1) \sum_{i=0}^k [\mathcal{T}_i]^\vee \otimes ([\mathcal{T}_{i+1}] - [\mathcal{T}_i]) \quad (5.9)$$

where  $[\mathcal{T}_{k+1}] := q_2[\mathcal{T}_0]$ .



**5.8.** We would like to express the above in terms of either  $[\mathcal{T}_0]$  or  $[\mathcal{T}_k]$  and the tautological line bundles:

$$l_i := [\mathcal{L}_i] = [\mathcal{T}_i] - [\mathcal{T}_{i-1}], \quad \forall i \in \{1, \dots, k\}$$

Relation (5.9) therefore gives:

$$\begin{aligned} [T_{\mathcal{F}} \mathfrak{U}_d^k] &= [T_{\mathcal{F}_0} \mathcal{M}_d] + [\mathcal{W}_0] \left( \frac{1}{l_1} + \dots + \frac{1}{l_k} \right) + (q_1 - 1) \sum_{i \leq j} \frac{l_i}{l_j} = \\ &= [T_{\mathcal{F}_k} \mathcal{M}_{d+k}] - q[\mathcal{W}_k]^\vee (l_1 + \dots + l_k) + (q_1 - 1) \sum_{i \leq j} \frac{l_i}{l_j} \end{aligned}$$

Letting  $p^-/p^+ : \mathfrak{U}_d^k \rightarrow \mathcal{M}_d/\mathcal{M}_{d+k}$  denote the standard projections to the first/last sheaf in the flag, the above relations can be written in the form:

$$[T_{\mathcal{F}} \mathfrak{U}_d^k] = p^{\pm*}([T\mathcal{M}]) \mp q^\varepsilon p^{\pm*}([\mathcal{W}]^\mp) (l_1^{\pm 1} + \dots + l_k^{\pm 1}) + (q_1 - 1) \sum_{i \leq j} \frac{l_i}{l_j} \quad (5.10)$$

where  $\varepsilon = 1$  or  $0$  depending on whether the sign is  $+$  or  $-$ , and we introduce the notation  $\mathcal{W}^+ = \mathcal{W}$  and  $\mathcal{W}^- = \mathcal{W}^\vee$ .

**5.9.** We can do the same computation as above for the variety:

$$\mathfrak{V}^k = \{\mathcal{F} \supset \mathcal{F}' \supset \mathcal{F}(-D)\} \subset \mathcal{M}_d \times \mathcal{M}_{d+k} \quad (5.11)$$

of (4.8). It is realized as a  $\mathbb{Z}/2\mathbb{Z}$ -fixed locus, and by analogy with (5.10), we have the following formula for the character of its tangent space:

$$[T\mathfrak{V}^k] = p^{\pm*}([T\mathcal{M}]) \mp q^\varepsilon p^{\pm*}([\mathcal{W}]^\mp) \cdot l^\pm + (q_1 - 1) \cdot l \cdot l^\vee \quad (5.12)$$

Here,  $l$  is the  $K$ -theory class of the tautological rank  $k$  vector bundle on  $\mathfrak{V}^k$ , whose fiber over  $\mathcal{F} \supset \mathcal{F}'$  is  $\mathrm{R}\Gamma(\mathcal{F}/\mathcal{F}')$ .

## 6. PUSH-FORWARDS

**6.1.** We will begin with some generalities. Consider a proper map of algebraic varieties  $\pi : X \rightarrow X_0$  and a  $K$ -theory class  $l \in K^1(X)$ . For any rational function  $P(u) \in K(X_0)(u)$ , we may consider the push-forward:

$$\pi_* (P(l)) \in K^*(X_0). \quad (6.1)$$

Now suppose that  $\pi$  factors as:

$$\pi : X_k \xrightarrow{\pi_k} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 \quad (6.2)$$

where each  $\pi_i$  is proper. We call the above a **tower**, and fix a class  $l_i \in K^*(X_i)$  for each  $i \in \{1, \dots, k\}$ . For any rational function  $P(u_1, \dots, u_k) \in K(X_0)(u_1, \dots, u_k)$ , we are interested in computing push-forwards of the form:

$$\pi_*(P(l_1, \dots, l_k)) \in K^*(X_0). \quad (6.3)$$

Though each  $l_i$  is a class on  $X_i$ , we view it as a class on  $X_k$  and suppress the obvious pull-back maps.

**6.2.** For a single proper map  $\pi : X \rightarrow X_0$ , let us assume that there exists a rational function  $f(u) \in K^*(X_0)(u)$  such that:

$$\pi_*(P(l)) = \int P(u)f(u)Du, \quad \forall P(u) \in K^*(X_0)(u) \quad (6.4)$$

where  $Du = \frac{du}{2\pi i}$  and the integral is taken over a contour that separates the poles of  $P$  from those of  $f$ . Specifying the contour is necessary, because we want it to only pick up residues at the poles of the rational function  $f$  and not at those of  $P$ . Relation (6.4) might seem like a formal artifice, but it comes up naturally in important situations:

*Example 6.3.* Suppose  $X = \mathbb{P}(\mathcal{V})$  for some vector bundle  $\mathcal{V}$  on  $X_0$ , and  $l = [\mathcal{O}(1)]$  is the tautological line bundle. Then  $f(u) = \Lambda(-\mathcal{V}, u)$  of subsection 2.4. When  $P(u) = u^k$  for  $k \in \mathbb{Z}$ , the push-forwards (6.1) are simply  $K$ -theoretic variants of the Segre classes of the bundle  $\mathcal{V}$ .

**6.4.** Take a tower as in (6.2) and suppose that for each proper map therein there exists a rational function  $f_i(u)$  fulfilling (6.4). Then by writing  $\pi = \pi_{1*} \dots \pi_{k*}$ , we can inductively conclude that:

$$\pi_*(P(l_1, \dots, l_k)) = \int P(u_1, \dots, u_k) f_1(u_1) \dots f_k(u_k) Du_1 \dots Du_k \quad (6.5)$$

where the contours of integration all separate the poles of the functions  $f_i$  from those of the rational function  $P$ , in such a way that we only pick up the residues at the former.

**6.5.** Now we will apply the whole setup above to our situation.

**Proposition 6.6.** *Let  $\pi^+, \pi^- : \mathfrak{U}_k \rightarrow \mathfrak{U}_{k-1}$  be the map which forgets the first/last sheaf in the flag, and let  $l$  be the tautological class that is forgotten. Then for any rational function  $P$  with coefficients in  $K_T^*(\mathfrak{U}_{k-1})$ , we have:*

$$\pi_*^\pm(P(l)) = \rho_\pm \int P(u) u^{-r\varepsilon} q_1^{k\varepsilon} \frac{\Lambda(\pm \mathcal{W}, uq^\varepsilon)}{1 - q_1^{\pm 1}} \prod_{i=1}^{k-1} \frac{1 - ul_i^{-1}}{1 - ul_i^{-1} q_1^{\pm 1}} Du \quad (6.6)$$

where recall that  $\varepsilon$  is 1 or 0 depending on whether the sign is + or -, while  $\rho_+ = (-q)^{-r} x_1^{-1} \dots x_r^{-1}$  and  $\rho_- = 1$ . The integral is over a contour which separates  $\text{Poles}(P) \cup \{0, \infty\}$  from any other possible poles of the integrand.

**Proof** By equivariant localization, we have:

$$\pi_*^\pm(P(l)) = \sum_{\lambda \in \mathfrak{U}_k^T} [\pi(\lambda)] \frac{P(l|_\lambda)}{\Lambda(T_\lambda \mathfrak{U}_k, 1)} \quad (6.7)$$

By (5.10), we see that:

$$[T_{\pi^\pm(\lambda)} \mathfrak{U}_{k-1}] - [T_\lambda \mathfrak{U}_k] = \pm q^\varepsilon p^{\pm*}([\mathcal{W}]^\mp) \cdot l^{\pm 1} + (1 - q_1) \sum_{i \leq k} \left( \frac{l}{l_i} \right)^{\pm 1} \quad (6.8)$$

where  $p^+, p^-$  maps a flag (5.7) onto its last/first sheaf. Therefore, we see that the RHS of (6.7) equals:

$$\sum_{\lambda_0 \in \mathfrak{U}_{k-1}^T} \frac{[\lambda_0]}{\Lambda(T_{\lambda_0} \mathfrak{U}_{k-1}, 1)} \sum_{\lambda \in \mathfrak{U}_k^T}^{\pi^\pm(\lambda) = \lambda_0} P(l|_\lambda) \prod_{w \in \mathcal{W}|_{\lambda_0}} (1 - w^{\pm 1} l|_\lambda^{\mp 1} q^{-\varepsilon})^{\pm 1} \prod_{i \leq k} \frac{1 - \frac{l_i |_\lambda^{\pm 1}}{l |_\lambda^{\pm 1}}}{1 - \frac{l_i |_\lambda^{\pm 1}}{q_1 l |_\lambda^{\pm 1}}}$$

Recall from (5.1) that the weights that appear in  $\mathcal{W}|_{\lambda_0}$  are the inner corners of the partition  $\lambda_0$  (with sign  $+$ ) and the outer corners of the partition  $\lambda_0$  (with sign  $-$ ). With that in mind, the above becomes:

$$\begin{aligned} \pi_*^\pm(P(l)) &= \sum_{\lambda_0 \in \mathfrak{U}_{k-1}^T} \frac{[\lambda_0]}{\Lambda(T_{\lambda_0} \mathfrak{U}_{k-1}, 1)} \sum_{\lambda \in \mathfrak{U}_k^T}^{\pi^\pm(\lambda) = \lambda_0} P(l|_\lambda) \\ &\quad \left[ \frac{\prod_{\square \text{ inner corner of } \lambda_0} (1 - \chi(\square)^{\pm 1} l|_\lambda^{\mp 1} q^{-\varepsilon})}{\prod_{\square \text{ outer corner of } \lambda_0} (1 - \chi(\square)^{\pm 1} l|_\lambda^{\mp 1} q^{-\varepsilon})} \right]^{\pm 1} q_1^{k\varepsilon} \prod_{i \leq k} \frac{1 - \frac{l|_\lambda}{l_i |_\lambda}}{1 - \frac{q_1^{\pm 1} l|_\lambda}{l_i |_\lambda}} \\ &= \sum_{\lambda_0 \in \mathfrak{U}_{k-1}^T} \frac{[\lambda_0]}{\Lambda(T_{\lambda_0} \mathfrak{U}_{k-1}, 1)} \sum_{\lambda \in \mathfrak{U}_k^T}^{\pi^\pm(\lambda) = \lambda_0} \rho_\pm \cdot P(l|_\lambda) l|_\lambda^{-r\varepsilon} \\ &\quad \left[ \frac{\prod_{\square \text{ inner corner of } \lambda_0} \left( 1 - \frac{q^\varepsilon l|_\lambda}{\chi(\square)} \right)}{\prod_{\square \text{ outer corner of } \lambda_0} \left( 1 - \frac{q^\varepsilon l|_\lambda}{\chi(\square)} \right)} \right]^{\pm 1} q_1^{k\varepsilon} \prod_{i \leq k} \frac{1 - \frac{l|_\lambda}{l_i |_\lambda}}{1 - \frac{q_1^{\pm 1} l|_\lambda}{l_i |_\lambda}} \end{aligned}$$

As  $\lambda$  goes over the the preimages under  $\pi^\pm$  of a fixed point  $\lambda_0$ , the character  $l|_\lambda$  goes over the residues of the function on the second line in the above relation. Therefore, we can write the above as:

$$\sum_{\lambda_0 \in \mathfrak{U}_{k-1}^T} \frac{[\lambda_0]}{\Lambda(T_{\lambda_0} \mathfrak{U}_{k-1}, 1)} \int \rho_\pm P(u) u^{-r\varepsilon} q_1^{k\varepsilon} \frac{\Lambda(\pm \mathcal{W}|_{\lambda_0}, u q^\varepsilon)}{1 - q_1^{\pm 1}} \prod_{i=1}^{k-1} \frac{1 - u l_i^{-1}}{1 - u l_i^{-1} q_1^{\pm 1}} Du$$

Going back to the localization formula, this gives precisely (6.6).  $\square$

**6.7.** The above proposition, together with the general framework of Subsection 6.4, allow us to prove the following more general result.

**Proposition 6.8.** *For a general projection  $p^\pm : \mathfrak{Z}_k \rightarrow \mathcal{M}_{d^\pm}$  and any rational function  $P(u_1, \dots, u_k)$  with coefficients in  $K$ , we have:*

$$p_*^\pm (P(l_1, \dots, l_k)) = \frac{\pm \rho_\pm^k (-q_1)^{(k-1)(1-\varepsilon)}}{q_1^{-1} - 1} \int P(u_1, \dots, u_k) (u_1 \dots u_k)^{-r\varepsilon} \frac{\prod_{i=1}^k \Lambda(\pm \mathcal{W}, u_i q^\varepsilon)}{u_1 \left( \frac{u_2}{u_1} - q \right) \dots \left( \frac{u_k}{u_{k-1}} - q \right) u_k^{-1}} \prod_{1 \leq i < j \leq k} \omega \left( \frac{u_j}{u_i} \right) Du_1 \dots Du_k \quad (6.9)$$

where the integrals are over  $k$  nested contours which separate the set  $\mathcal{P} = \text{Poles}(P) \cup \{0, \infty\}$  from all other poles in the integrand. The order of the contours is the following: depending on whether the sign is  $+$  or  $-$ , then we let  $u_1$  or  $u_k$  be the contour closest to  $\mathcal{P}$  and then continue in order.

**Proof** The maps  $p^\pm : \mathfrak{Z}^k \rightarrow \mathcal{M}$  can be written as projective towers:

$$p^+ : \mathfrak{Z}^k \xrightarrow{p^{1+}} \mathfrak{Z}^{k-1} \xrightarrow{p^{2+}} \dots \xrightarrow{p^{(k-1)+}} \mathfrak{Z}^1 \xrightarrow{p^{k+}} \mathcal{M}_{d^+}$$

$$p^- : \mathfrak{Z}^k \xrightarrow{p^{k-}} \mathfrak{Z}^{k-1} \xrightarrow{p^{(k-1)-}} \dots \xrightarrow{p^{2-}} \mathfrak{Z}^1 \xrightarrow{p^{1-}} \mathcal{M}_{d^-}$$

where  $p^{i\pm}$  is the map that forgets the inclusion marked by  $i$ . Let us first do the case of  $+$ . For each  $i \in \{1, \dots, k-1\}$ , consider the following vector bundle on  $\mathcal{M}_{d_i} \times \mathfrak{Z}^{k-i}$  as in subsection 4.6:

$$\tilde{E}|_{\mathcal{F}^i \times (\mathcal{F}^{i+1} \supset_x \dots \supset_x \mathcal{F}^{k+1})} = \text{Ext}^1(\mathcal{F}^{i+1}, \mathcal{F}^i(-\infty))$$

As we have seen in Proposition 4.7, this bundle has a section which vanishes schemetically on the locus:

$$e(\tilde{E}) = \{\mathcal{F}^i \supset \mathcal{F}^{i+1} \supset_x \dots \supset_x \mathcal{F}^{k+1}\} = \{\mathcal{F}^i \supset_x \mathcal{F}^{i+1} \supset_x \dots \supset_x \mathcal{F}^{k+1}\} \cup \bigcup \overline{\{\mathcal{F}^i \supset_y \mathcal{F}^{i+1} \supset_x \dots \supset_x \mathcal{F}^{k+1}, y \neq x\}} = [\mathfrak{Z}^{k-i+1}] + [\mathfrak{U}_{d_i}^2 \times \mathfrak{U}_{d_{i+1}}^1 \mathfrak{Z}^{k-i}]$$

Letting

$$\pi : \mathcal{M}_{d_i} \times \mathfrak{Z}^{k-i} \rightarrow \mathfrak{Z}^{k-i}, \quad q : \mathfrak{U}_{d_i}^2 \times \mathfrak{U}_{d_{i+1}}^1 \mathfrak{Z}^{k-i} \rightarrow \mathfrak{Z}^{k-i}$$

denote the standard projections, the above allows us to compute  $p_*^{i+}$ :

$$p_*^{i+}(Q(l_i)) = \pi_* \left( e(\tilde{E}) \cdot Q(l_i) \right) - q_*(Q(l_i))$$

for any  $Q(u) \in K_T^*(\mathfrak{Z}^{k-i})(u)$ . The pushforwards  $\pi_*(e(\tilde{E}) \cdot \dots)$  and  $q_*(\dots)$  are computed exactly as in Proposition 6.6 (for  $k=1$  and  $2$ , respectively), so we see that:

$$p_*^{i+}(Q(l_i)) = \rho_+ \int \frac{Q(u)u^{-r}}{q_1^{-1} - 1} \left( \Lambda(\mathcal{W}^{i+1}, uq) - q_1 \Lambda(\mathcal{W}^{i+2}, uq) \frac{1 - ul_{i+1}^{-1}}{1 - ul_{i+1}^{-1}q_1} \right) Du$$

One obtains a similar relation in the case of  $-$ :

$$p_*^{i-}(Q(l_i)) = \rho_- \int \frac{Q(u)}{1 - q_1^{-1}} \left( \Lambda(-\mathcal{W}^i, u) - \Lambda(-\mathcal{W}^{i-1}, u) \frac{1 - ul_{i-1}^{-1}}{1 - ul_{i-1}^{-1}q_1^{-1}} \right) Du$$

From the fixed point formula, if  $\mathcal{F} \supset \mathcal{F}'$ , then we have:

$$\begin{aligned} [\mathcal{W}] &= [\mathcal{W}'] + (q_1 - 1)(q_2 - 1) \cdot l \implies \\ \implies \Lambda(\pm \mathcal{W}, u) &= \Lambda(\pm \mathcal{W}', u) \cdot \omega\left(\frac{u}{l}\right)^{\pm 1} \end{aligned} \quad (6.10)$$

Together with the formula  $\omega(q/x) = \omega(x)$ , the above gives us:

$$\begin{aligned} p_*^{i+}(Q(l_i)) &= \rho_+ \int Q(u)u^{-r} \Lambda(\mathcal{W}^{i+2}, uq) \cdot \frac{\frac{l_{i+1}}{u}}{\frac{l_{i+1}}{u} - q} \omega\left(\frac{l_{i+1}}{u}\right) Du \\ p_*^{i-}(Q(l_i)) &= -\rho_- q_1 \int Q(u) \Lambda(-\mathcal{W}^{i-1}, u) \cdot \frac{\frac{u}{l_{i-1}}}{\frac{u}{l_{i-1}} - q} \omega\left(\frac{u}{l_{i-1}}\right) Du \end{aligned}$$

Now we can iterate (6.10), and obtain:

$$\begin{aligned} p_*^{i+}(Q(l_i)) &= \rho_+ \int Q(u)u^{-r} \Lambda(\mathcal{W}^k, uq) \cdot \frac{\frac{l_{i+1}}{u}}{\frac{l_{i+1}}{u} - q} \prod_{j>i} \omega\left(\frac{l_j}{u}\right) Du \\ p_*^{i-}(Q(l_i)) &= -\rho_- q_1 \int Q(u) \Lambda(-\mathcal{W}^0, u) \cdot \frac{\frac{u}{l_{i-1}}}{\frac{u}{l_{i-1}} - q} \prod_{j<i} \omega\left(\frac{u}{l_j}\right) Du \end{aligned}$$

and then the desired relation follows from (6.5).

□

**6.9.** A simple application of the above Proposition allows us to prove the following result, which implies Theorem 3.10:

**Theorem 6.10.** *For any Laurent polynomial  $m(l_1, \dots, l_k)$ , the operator  $a_m^\pm$  of (4.2) acts on  $K$  via the shuffle element:*

$$\begin{aligned} P_m^- &:= \frac{(-q_1)^{(k-1)}}{(q_1^{-1} - 1)(q^{-1} - 1)^k} \cdot \text{Sym} \left[ \frac{m(z_{-1}, \dots, z_{-k})}{z_{-1} \left( \frac{z_{-2}}{z_{-1}} - q \right) \dots \left( \frac{z_{-k}}{z_{-(k-1)}} - q \right) z_{-k}^{-1}} \prod_{1 \leq i < j \leq k} \omega\left(\frac{z_{-j}}{z_{-i}}\right) \right] \\ P_m^+ &:= \frac{[(-q)^r x_1 \dots x_r]^{-k}}{(q_1^{-1} - 1)(q - 1)^k} \cdot \text{Sym} \left[ \frac{m(z_k, \dots, z_1) \cdot (z_{-1} \dots z_{-k})^{-r}}{z_1^{-1} \left( \frac{z_1}{z_2} - q \right) \dots \left( \frac{z_{k-1}}{z_k} - q \right) z_k} \prod_{1 \leq i < j \leq k} \omega\left(\frac{z_i}{z_j}\right) \right] \end{aligned}$$

**Proof** By (4.2), we have:

$$a_m^\pm \left( \prod_{s \in S} \Lambda(\mathcal{T}_d, s) \right) = p_*^\pm \left( m(l_1, \dots, l_k) \prod_{s \in S} \Lambda(p^{\mp*} \mathcal{T}_d, s) \right)$$

Comparing the various tautological sheaves on the variety  $\mathfrak{Z}_k$ , we can write:

$$\Lambda(p^{\mp*}(\mathcal{T}), s) = \Lambda(p^{\pm*}(\mathcal{T}), s) \prod_{i=1}^k \left( 1 - \frac{s}{l_i} \right)^{\mp 1}$$

and therefore:

$$a_m^\pm \left( \prod_{s \in S} \Lambda(\mathcal{T}_d, s) \right) = \prod_{s \in S} \Lambda(\mathcal{T}_{d \pm k}, s) \cdot p_*^\pm \left( m(l_1, \dots, l_k) \prod_{s \in S} \prod_{i=1}^k \left( 1 - \frac{s}{l_i} \right)^{\mp 1} \right)$$

which according to (6.9) equals:

$$\begin{aligned} & \frac{\pm \rho_\pm^k (-q_1)^{(k-1)(1-\varepsilon)}}{q_1^{-1} - 1} \prod_{s \in S} \Lambda(\mathcal{T}_{d \pm k}, s) \int \frac{m(u_1, \dots, u_k) \cdot (u_1 \dots u_k)^{-r\varepsilon}}{\prod_{s \in S} \prod_{i=1}^k \left( 1 - \frac{s}{u_i} \right)^{\pm 1}} \\ & \frac{\prod_{i=1}^k \Lambda(\pm \mathcal{W}_{d \pm k}, u_i q^\varepsilon)}{u_1 \left( \frac{u_2}{u_1} - q \right) \dots \left( \frac{u_k}{u_{k-1}} - q \right) u_k^{-1}} \prod_{1 \leq i < j \leq k} \omega \left( \frac{u_j}{u_i} \right) Du_1 \dots Du_k \end{aligned}$$

Comparing this with the definition of the shuffle algebra action in (3.10) gives us the desired result. The reason for the different formulas in the cases  $+/-$  is that the order of the contours prescribed in Proposition 2.5 is different for the two values of the sign. Concretely, when the sign is  $+$  we need to invert the order of the variables, so that we match the order of the contours with the normal ordering convention of Subsection 3.7.

□

## 7. THE VECTOR BUNDLE $E$ AND THE VARIETIES $\mathfrak{V}^k$

**7.1.** We will now prove Proposition 4.11, namely compute those shuffle elements which act on  $K$  like the geometric operators  $b_k^\pm$  and  $c_k^\pm$ . We will compute them by equivariant localization:

$$b_k^\pm \left( \prod_{s \in S} \Lambda(\mathcal{T}, s) \right) = \sum_{\lambda_+, \lambda_- \in \mathcal{M}^T} [\lambda_\pm] \frac{\Lambda(E_{\lambda_-, \lambda_+}, 1)}{\Lambda(T_{\lambda_\pm} \mathcal{M}, 1) \Lambda(T_{\lambda_\mp} \mathcal{M}, 1)} \prod_{s \in S} \Lambda(\mathcal{T}_{\lambda^\mp}, s) \quad (7.1)$$

At the very end of subsection 5.5, we have remarked that the character of  $E_{\lambda_-, \lambda_+}$  contains the trivial character (and therefore the above  $\Lambda(E, 1)$  vanishes) unless  $\lambda_- \leq \lambda_+$ . Therefore, we need only take such pairs of fixed points in the above sum. Using (5.3), we see that:

$$[E_{\lambda_-, \lambda_+}] - [T_{\lambda_\mp} \mathcal{M}] = \pm q^\varepsilon [\mathcal{W}_{\lambda_\mp}]^\mp \otimes ([\mathcal{T}_{\lambda_+}]^\pm - [\mathcal{T}_{\lambda_-}]^\pm),$$

which can be rewritten as:

$$[E_{\lambda_-, \lambda_+}] - [T_{\lambda_{\mp}} \mathcal{M}] = \pm q^\varepsilon [\mathcal{W}_{\lambda_{\pm}}]^\mp \otimes \sum_{\square \in \lambda_+ \setminus \lambda_-} \chi(\square)^{\pm 1 + (q_1 - 1)(q_2 - 1)} \sum_{\square, \square' \in \lambda_+ \setminus \lambda_-} \frac{\chi(\square)}{\chi(\square')}$$

Therefore, (7.1) yields:

$$\begin{aligned} b_k^\pm \left( \prod_{s \in S} \Lambda(\mathcal{T}, s) \right) &= \sum_{\lambda_- \subset \lambda_+} \frac{[\lambda_\pm]}{\Lambda(T_{\lambda_\pm} \mathcal{M}, 1)} \prod_{s \in S} \Lambda(\mathcal{T}_{\lambda_\pm}, s) \prod_{\substack{w \in \mathcal{W}_{\lambda_\pm} \\ \square \in \lambda_+ \setminus \lambda_-}} (1 - q^{-\varepsilon} w^{\pm 1} \chi(\square)^{\mp 1})^{\pm 1} \\ &\quad \prod_{\square, \square' \in \lambda_+ \setminus \lambda_-} \frac{\left(1 - \frac{\chi(\square')}{\chi(\square)}\right) \left(1 - \frac{\chi(\square')}{q \chi(\square)}\right)}{\left(1 - \frac{\chi(\square')}{q_1 \chi(\square)}\right) \left(1 - \frac{\chi(\square')}{q_2 \chi(\square)}\right)} \cdot \prod_{\square \in \lambda_+ \setminus \lambda_-} \prod_{s \in S} \left(1 - \frac{s}{\chi(\square)}\right)^{\mp 1} = \\ &= \rho_\pm^k \sum_{\lambda_- \subset \lambda_+} \frac{[\lambda_\pm]}{\Lambda(T_{\lambda_\pm} \mathcal{M}, 1)} \prod_{s \in S} \Lambda(\mathcal{T}_{\lambda_\pm}, s) \end{aligned} \quad (7.2)$$

$$\prod_{\square \in \lambda_+ \setminus \lambda_-} \frac{\chi(\square)^{-r\varepsilon}}{\prod_{s \in S} \left(1 - \frac{s}{\chi(\square)}\right)^{\pm 1}} \left[ \frac{\prod_{\text{corner of } \lambda_\pm}^{\square' \text{ inner}} \left(1 - \frac{\chi(\square)}{q^\varepsilon \chi(\square')}\right)}{\prod_{\text{corner of } \lambda_\pm}^{\square' \text{ outer}} \left(1 - \frac{\chi(\square)}{q^\varepsilon \chi(\square')}\right)} \right]^{\pm 1} \prod_{\square, \square' \in \lambda_+ \setminus \lambda_-} \omega\left(\frac{\chi(\square')}{\chi(\square)}\right)$$

**7.2.** As was mentioned before, the above sum goes over all  $\lambda_- \leq \lambda_+$  with  $|\lambda_+ \setminus \lambda_-| = k$ . Therefore, if we fix the partition  $\lambda_+$  (or  $\lambda_-$ ), the sum goes over all the ways to remove (or add, respectively)  $k$  non-ordered boxes  $\square$  from this partition. We claim that the corresponding  $\chi(\square)$  are precisely the poles of a rational function, in that the above relation becomes:

$$\begin{aligned} &\rho_\pm^k \sum_{\lambda_\pm} \frac{[\lambda_\pm]}{\Lambda(T_{\lambda_\pm} \mathcal{M}, 1)} \prod_{s \in S} \Lambda(\mathcal{T}_{\lambda_\pm}, s) \cdot \left( \frac{1 - q}{(1 - q_1)(1 - q_2)} \right)^k \int Du_1 \dots Du_k \\ &\prod_{i=1}^k \frac{u_i^{-r\varepsilon}}{\prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{\pm 1}} \left[ \frac{\prod_{\text{corner of } \lambda_\pm}^{\square' \text{ inner}} \left(1 - \frac{\chi(\square)}{q^\varepsilon \chi(\square')}\right)}{\prod_{\text{corner of } \lambda_\pm}^{\square' \text{ outer}} \left(1 - \frac{\chi(\square)}{q^\varepsilon \chi(\square')}\right)} \right]^{\pm 1} \prod_{1 \leq i \neq j \leq k} \omega\left(\frac{u_j}{u_i}\right) \end{aligned}$$

where the contours separate  $S \cup \{0, \infty\}$  from any other poles in the integrand. This can be seen simply by performing the integrals one by one, and accounting for the residues. Doing so will also reveal a slight inaccuracy in the above formula: it does not count each configuration of added/removed boxes once (as does (7.2)), but it counts each of them with the number of ways in which we can build up the configuration  $\lambda_\mp$  from  $\lambda_\pm$  in such a way that intermediary steps are still partitions. To correct this, we need to take the normal-ordered integral:

$$b_k^\pm \left( \prod_{s \in S} \Lambda(\mathcal{T}, s) \right) = \left[ \frac{\rho_\pm(1 - q)}{(1 - q_1)(1 - q_2)} \right]^k \sum_{\lambda_\pm} \frac{[\lambda_\pm]}{\Lambda(T_{\lambda_\pm} \mathcal{M}, 1)} \prod_{s \in S} \Lambda(\mathcal{T}_{\lambda_\pm}, s) : \int :$$

$$(u_1 \dots u_k)^{-r\varepsilon} \prod_{i=1}^k \Lambda(\pm \mathcal{W}_{\lambda_{\pm}}, u_i q^{\varepsilon}) \prod_{1 \leq i \neq j \leq k} \omega\left(\frac{u_j}{u_i}\right) \prod_{i=1}^k \prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{\mp 1} Du_1 \dots Du_k$$

De-localizing, we see that:

$$b_k^{\pm} \left( \prod_{s \in S} \Lambda(\mathcal{T}, s) \right) = \left[ \frac{\rho_{\pm}(1-q)}{(1-q_1)(1-q_2)} \right]^k \prod_{s \in S} \Lambda(\mathcal{T}, s) : \int : Du_1 \dots Du_k$$

$$(u_1 \dots u_k)^{-r\varepsilon} \prod_{i=1}^k \Lambda(\pm \mathcal{W}, u_i q^{\varepsilon}) \prod_{1 \leq i \neq j \leq k} \omega\left(\frac{u_j}{u_i}\right) \prod_{i=1}^k \prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{\mp 1}$$

whence (4.9).

**7.3.** To prove (4.10), we need to play the same as before with the operator  $c_k^{\pm}$ . Localization gives us:

$$c_k^{\pm} \left( \prod_{s \in S} \Lambda(\mathcal{T}, s) \right) = \sum_{\lambda_- \leq \lambda_+} \frac{[\lambda_{\pm}]}{\Lambda(T_{(\lambda_-, \lambda_+)}^k, 1)} \prod_{s \in S} \Lambda(\mathcal{T}_{\lambda^{\mp}}, s)$$

We can use relation (5.12) to compute:

$$[T\mathfrak{V}^k] - p^{\pm*}([T\mathcal{M}]) = \mp q^{\varepsilon} p^{\pm*}([\mathcal{W}]^{\mp}) \sum_{\square \in \lambda_+ \setminus \lambda_-} \chi_{\square}^{\pm} + (q_1 - 1) \sum_{\square, \square' \in \lambda_+ \setminus \lambda_-} \frac{\chi(\square)}{\chi(\square')}$$

This differs from Subsection 7.1 only in the coefficient in front of the last term ( $q_1 - 1$  instead of  $(q_1 - 1)(q_2 - 1)$ ), and the whole argument therein carries through:

$$c_k^{\pm} \left( \prod_{s \in S} \Lambda(\mathcal{T}, s) \right) = \left[ \frac{\rho_{\pm}}{1 - q_1^{-1}} \right]^k \prod_{s \in S} \Lambda(\mathcal{T}, s) : \int : Du_1 \dots Du_k$$

$$(u_1 \dots u_k)^{-r\varepsilon} \prod_{i=1}^k \Lambda(\pm \mathcal{W}, u_i q^{\varepsilon}) \prod_{1 \leq i \neq j \leq k} \frac{u_j - u_i}{u_j - q_1^{-1} u_i} \prod_{i=1}^k \prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{\mp 1}$$

This proves the (4.10).

## 8. APPENDIX

**8.1.** In the remainder of this paper, we will prove Theorem 3.6, by showing that the action of  $\mathcal{A}$  on  $K$  given by (3.9) and (3.10) respects the relations (3.2), (3.6) and (3.7). To show the first of these, it is enough to prove it for:

$$P(z_{\pm 1}, \dots, z_{\pm k}) = \text{Sym} \left[ z_{\pm 1}^{\lambda_1} \dots z_{\pm k}^{\lambda_k} \prod_{1 \leq i < j \leq k} \omega \left( \left( \frac{z_{\pm i}}{z_{\pm j}} \right)^{\pm 1} \right) \right] \quad (8.1)$$



$$Q(z_{\pm 1}, \dots, z_{\pm l}) = \text{Sym} \left[ z_{\pm 1}^{\mu_1} \dots z_{\pm l}^{\mu_l} \prod_{1 \leq i < j \leq l} \omega \left( \left( \frac{z_{\pm i}}{z_{\pm j}} \right)^{\pm 1} \right) \right]$$

By iterating (3.10), and taking into account the definition of the time ordered integral, we see that:

$$\begin{aligned} P(z_{\pm 1}, \dots, z_{\pm k}) * \left[ Q(z_{\pm 1}, \dots, z_{\pm l}) * \prod_{s \in S} \Lambda(\mathcal{T}_d, s) \right] &= [\pm(q^{\pm 1} - 1)]^{k+l} \prod_{s \in S} \Lambda(\mathcal{T}_{d \pm k \pm l}, s) \\ &\int u_1^{\lambda_1} \dots u_k^{\lambda_k} u_{k+1}^{\mu_1} \dots u_{k+l}^{\mu_l} \prod_{1 \leq i < j \leq k} \omega \left( \left( \frac{z_{\pm i}}{z_{\pm j}} \right)^{\pm 1} \right) \prod_{k+1 \leq i < j \leq k+l} \omega \left( \left( \frac{z_{\pm i}}{z_{\pm j}} \right)^{\pm 1} \right) \\ &\prod_{i=1}^k \prod_{j=k+1}^{k+l} \omega \left( \frac{u_j q^\varepsilon}{u_i} \right) \prod_{i=1}^k \frac{\Lambda(\pm \mathcal{W}, u_i q^\varepsilon)}{\prod_{s \in S} \left( 1 - \frac{s}{u_i} \right)^{\pm 1}} \prod_{j=k+1}^{k+l} \frac{\Lambda(\pm \mathcal{W}, u_j q^\varepsilon)}{\prod_{s \in S} \left( 1 - \frac{s}{u_j} \right)^{\pm 1}} D u_1 \dots D u_{k+l} \end{aligned}$$

The latter is precisely the time-ordered integral that yields:

$$[P(z_{\pm 1}, \dots, z_{\pm k}) * Q(z_{\pm 1}, \dots, z_{\pm l})] * \prod_{s \in S} \Lambda(\mathcal{T}_d, s)$$

which proves that (3.2) is satisfied.

**8.2.** We will now prove relation (3.6), once again for  $P$  of the form (8.1):

$$\begin{aligned} P(z_{\pm 1}, \dots, z_{\pm k}) * h^{\pm'}(z) * \prod_{s \in S} \Lambda(\mathcal{T}_d, s) &= [\pm(q^{\pm 1} - 1)]^k \frac{\Lambda(\mathcal{W}_{d \pm k}, zq)}{\Lambda(\mathcal{W}_{d \pm k}, z)} \prod_{s \in S} \Lambda(\mathcal{T}_{d \pm k}, s) \\ &\int u_1^{\lambda_1} \dots u_k^{\lambda_k} \prod_{i=1}^k \left[ \frac{\Lambda(\pm \mathcal{W}, u_i q^\varepsilon)}{\prod_{s \in S} \left( 1 - \frac{s}{u_i} \right)^{\pm 1}} \left[ \frac{(u_i - z/q)(u_i - zq_1)(u_i - zq_2)}{(u_i - zq)(u_i - z/q_1)(u_i - z/q_2)} \right]^{\pm 1} \right] D u_1 \dots D u_k \end{aligned}$$

The above integral is the time-ordered integral which gives:

$$h^{\pm'}(z) * \left[ P(z_{\pm 1}, \dots, z_{\pm k}) \prod_{i=1}^k \Omega \left( \frac{z_{\pm i}}{z} \right)^{\pm 1} \right] * \prod_{s \in S} \Lambda(\mathcal{T}_d, s)$$

which proves (3.6).

**8.3.** Finally, we will prove the commutation relation (3.7), so let us take:

$$\begin{aligned} P(z_1, \dots, z_k) &= \text{Sym} \left[ z_1^{\lambda_1} \dots z_k^{\lambda_k} \prod_{1 \leq i < j \leq k} \omega \left( \frac{z_i}{z_j} \right) \right] \\ Q(z_{-1}, \dots, z_{-l}) &= \text{Sym} \left[ z_{-1}^{\mu_1} \dots z_{-l}^{\mu_l} \prod_{1 \leq i < j \leq l} \omega \left( \frac{z_{-j}}{z_{-i}} \right) \right] \end{aligned} \quad (8.2)$$

By definition, we have:

$$P(z_1, \dots, z_k) * Q(z_{-1}, \dots, z_{-l}) * \prod_{s \in S} \Lambda(\mathcal{T}_d, s) = (q-1)^k (1-q^{-1})^l \prod_{s \in S} \Lambda(\mathcal{T}_{d+k-l}, s)$$

$$\int u_1^{\lambda_1} \dots u_k^{\lambda_k} v_1^{\mu_1} \dots v_l^{\mu_l} \prod_{1 \leq i < j \leq k} \omega\left(\frac{u_i}{u_j}\right) \prod_{1 \leq i < j \leq l} \omega\left(\frac{v_j}{v_i}\right) \prod_{i=1}^k \prod_{j=1}^l \omega^{-1}\left(\frac{v_j}{u_i}\right)$$

$$\prod_{i=1}^k \left( \Lambda(\mathcal{W}, u_i q) \prod_{s \in S} \left(1 - \frac{s}{u_i}\right)^{-1} \right) \prod_{j=1}^l \left( \Lambda(-\mathcal{W}, v_j) \prod_{s \in S} \left(1 - \frac{s}{v_j}\right) \right) Du_1 \dots Du_k Dv_1 \dots Dv_l$$

Recall our convention: the contour of  $u_1$  is the closest to  $\text{Poles}(\pm \mathcal{W})$ , while that of  $v_l$  is the one farthest away. When we compute  $Q * P * \prod_{s \in S} \Lambda(\mathcal{T}_d, s)$ , we obtain the exact same integrand, but with  $v_1$  the closest to  $\text{Poles}(\pm \mathcal{W})$  and  $u_k$  the farthest away. Therefore, in order to compute how the commutator (3.7) acts on  $\prod_{s \in S} \Lambda(\mathcal{T}_d, s)$ , we need to see what residues we pick up as we move the  $u$  contours over the  $v$  contours.

Looking at the poles of  $\omega^{-1}$ , we see that there are two types of residues we can pick up: when  $u_i = v_j/q$  or  $u_i = v_j$  for some  $i, j$ . In the **first case**, let us relabel the colliding variables  $u_i = w = v_j/q$ . The residue we pick up is:

$$\frac{(q_1 - 1)(q_2 - 1)}{q - 1} \int Dv_1 \dots Dv_{j-1} Du_1 \dots Du_{i-1} Dw Du_{i+1} \dots Du_k Dv_{j+1} \dots Dv_l$$

$$u_1^{\lambda_1} \dots w^{\lambda_i} \dots u_k^{\lambda_k} v_1^{\mu_1} \dots (wq)^{\mu_j} \dots v_l^{\mu_l} \prod_{1 \leq i' < j' \leq k}^{i' \neq i \neq j'} \omega\left(\frac{u_{i'}}{u_{j'}}\right) \prod_{1 \leq i' < j' \leq l}^{i' \neq j \neq j'} \omega\left(\frac{v_{j'}}{v_{i'}}\right)$$

$$\prod_{j' \neq j}^{i' \neq i} \omega^{-1}\left(\frac{v_{j'}}{u_{i'}}\right) \prod_{i' \neq i} \left[ \Lambda(\mathcal{W}, u_{i'} q) \prod_{s \in S} \left(1 - \frac{s}{u_{i'}}\right)^{-1} \right] \prod_{i' > i} \Omega\left(\frac{u_{i'}}{w}\right)$$

$$\prod_{j' \neq j} \left[ \Lambda(-\mathcal{W}, v_{j'}) \prod_{s \in S} \left(1 - \frac{s}{v_{j'}}\right) \right] \prod_{j' > j} \Omega\left(\frac{wq}{v_{j'}}\right) \cdot \prod_{s \in S} \frac{1 - \frac{s}{wq}}{1 - \frac{s}{w}}$$

The order of the contours implies there are no poles inside the  $w$  contour, and therefore the above integral is 0. In the **second case**, let us relabel the colliding variables  $u_i = v_j = w$ , and the residue we pick up is:

$$\frac{(1 - q_1)(1 - q_2)}{1 - q} \int Dv_1 \dots Dv_{j-1} Du_1 \dots Du_{i-1} Dw Du_{i+1} \dots Du_k Dv_{j+1} \dots Dv_l$$

$$u_1^{\lambda_1} \dots w^{\lambda_i} \dots u_k^{\lambda_k} v_1^{\mu_1} \dots w^{\mu_j} \dots v_l^{\mu_l} \prod_{i' < j'}^{i' \neq i \neq j'} \omega\left(\frac{u_{i'}}{u_{j'}}\right) \prod_{i' < j'}^{i' \neq j \neq j'} \omega\left(\frac{v_{j'}}{v_{i'}}\right) \prod_{j' \neq j}^{i' \neq i} \omega^{-1}\left(\frac{v_{j'}}{u_{i'}}\right)$$

$$\frac{\Lambda(\mathcal{W}, wq)}{\Lambda(\mathcal{W}, w)} \prod_{i' \neq i} \frac{\Lambda(\mathcal{W}, u_{i'} q)}{\prod_{s \in S} \left(1 - \frac{u_{i'}}{s}\right)} \prod_{i' < i} \Omega\left(\frac{w}{u_{i'}}\right) \prod_{j' \neq j} \frac{\Lambda(-\mathcal{W}, v_{j'})}{\prod_{s \in S} \left(1 - \frac{v_{j'}}{s}\right)^{-1}} \prod_{j' < j} \Omega\left(\frac{v_{j'}}{w}\right)$$

We can move the  $w$  contour toward 0 and  $\infty$  without being obstructed by any poles, and therefore:

$$\begin{aligned}
[P, Q] * \prod_{s \in S} \Lambda(\mathcal{T}_d, s) &= (q-1)^{k-1} (1-q^{-1})^{l-1} \cdot \alpha \cdot \prod_{s \in S} \Lambda(\mathcal{T}_{d+k-l}, s) \\
\text{Res} \frac{\Lambda(\mathcal{W}, wq)}{\Lambda(\mathcal{W}, w)} \sum_{i=1}^k \sum_{j=1}^l \int & Dv_1 \dots Dv_{j-1} Du_1 \dots Du_{i-1} Du_{i+1} \dots Du_k Dv_{j+1} \dots Dv_l \\
& u_1^{\lambda_1} \dots u_i^{\lambda_i} \dots u_k^{\lambda_k} v_1^{\mu_1} \dots v_l^{\mu_l} \prod_{i' < j'}^{i' \neq i \neq j'} \omega\left(\frac{u_{i'}}{u_{j'}}\right) \prod_{i' < j'}^{i' \neq j \neq j'} \omega\left(\frac{v_{j'}}{v_{i'}}\right) \prod_{j' \neq j}^{i' \neq i} \omega^{-1}\left(\frac{v_{j'}}{u_{i'}}\right) \\
& \prod_{i' \neq i} \frac{\Lambda(\mathcal{W}, u_{i'} q)}{\prod_{s \in S} \left(1 - \frac{s}{u_{i'}}\right)} \prod_{i' < i} \Omega\left(\frac{w}{u_{i'}}\right) \prod_{j' \neq j} \frac{\Lambda(-\mathcal{W}, v_{j'})}{\prod_{s \in S} \left(1 - \frac{s}{v_{j'}}\right)^{-1}} \prod_{j' < j} \Omega\left(\frac{v_{j'}}{w}\right)
\end{aligned}$$

where Res refers to the residue at 0 minus the residue at  $\infty$  in the variable  $w$ . We are still not done, because our desired result must have all the  $v$  integrals to the left of the  $u$  integrals. So we need to iterate the above procedure, and this will pick up more residues along the same lines as the above:

$$\begin{aligned}
[P, Q] * \prod_{s \in S} \Lambda(\mathcal{T}_d, s) &= \sum_{a=1}^{\min(k, l)} (q-1)^{k-a} (1-q^{-1})^{l-a} \alpha^a \cdot \prod_{s \in S} \Lambda(\mathcal{T}_{d+k-l}, s) \\
\text{Res} \prod_{x=1}^a \frac{\Lambda(\mathcal{W}, w_x q)}{\Lambda(\mathcal{W}, w_x)} \sum_{\substack{1 \leq i_1, \dots, i_a \leq k \\ 1 \leq j_1 < \dots < j_a \leq l}}^{i_x \neq i_y} \int & Dv_1 \dots Dv_{j_1, \dots, a} \dots Dv_l Du_1 \dots Du_{i_1, \dots, a} \dots Du_k \\
& \prod_{x=1}^a w_x^{\lambda_{i_x} + \mu_{j_x}} \prod_{i \neq i_1, \dots, a} u_i^{\lambda_i} \prod_{j \neq j_1, \dots, a} v_j^{\mu_j} \prod_{i < i'}^{i, i' \neq i_1, \dots, a} \omega\left(\frac{u_i}{u_{i'}}\right) \prod_{j < j'}^{j, j' \neq j_1, \dots, a} \omega\left(\frac{v_j}{v_{j'}}\right) \\
& \prod_{j \neq j_1, \dots, a}^{i \neq i_1, \dots, a} \omega^{-1}\left(\frac{v_j}{u_i}\right) \prod_{i \neq i_1, \dots, a} \frac{\Lambda(\mathcal{W}, u_i q)}{\prod_{s \in S} \left(1 - \frac{s}{u_i}\right)} \prod_{j \neq j_1, \dots, a} \frac{\Lambda(-\mathcal{W}, v_j)}{\prod_{s \in S} \left(1 - \frac{s}{v_j}\right)^{-1}} \\
& \prod_{x=1}^a \left[ \prod_{1 \leq i < i_x}^{i \neq i_1, \dots, a} \Omega\left(\frac{w_x}{u_i}\right) \prod_{1 \leq j < j_x}^{j \neq j_1, \dots, a} \Omega\left(\frac{v_j}{w_x}\right) \right] \prod_{1 \leq x < y \leq a}^{i_x > i_y} \Omega\left(\frac{w_x}{w_y}\right) \quad (8.3)
\end{aligned}$$

where Res refers to the residue at 0 minus the residue at  $\infty$  in the variables  $w_1, \dots, w_a$ , computed in this order. The above equals precisely:

$$\begin{aligned}
& \sum_{a=1}^{\min(k, l)} \alpha^a \prod_{s \in S} \Lambda(\mathcal{T}_{d+k-l}, s) \cdot \text{Res} \prod_{x=1}^a \frac{\Lambda(\mathcal{W}, w_x q)}{\Lambda(\mathcal{W}, w_x)} \prod_{1 \leq x \neq y \leq a} \omega^{-1}\left(\frac{w_x}{w_y}\right) \\
& \frac{Q'(z_{-1}, \dots, z_{-(l-a)}, w_1, \dots, w_a)}{\prod_{j=1}^{l-a} \prod_{x=1}^a \omega\left(\frac{z_{-j}}{w_x}\right)} * \frac{P(z_1, \dots, z_{k-a}, w_1, \dots, w_a)}{\prod_{i=1}^{k-a} \prod_{x=1}^a \omega\left(\frac{w_x}{z_i}\right)} * \prod_{s \in S} \Lambda(\mathcal{T}_d, s)
\end{aligned}$$

where the difference between  $Q'$  and  $Q$  is that in (8.2) we only keep those permutations which keep  $w_1, \dots, w_a$  in order. This is in tune with the definition of the normal ordered residue  $: \text{Res} :$ , hence the desired conclusion of (3.7).

**8.4.** For general  $(k, d) = (na, nb) \in \mathbb{Z}^2$  with  $\gcd(a, b) = 1$ , the shuffle elements  $P_{k,d}$  we introduced in Subsection 3.11 have been shown in [6] to be given by:

$$P_{k,d} = c_{k,d} \sum_{x=0}^{n-1} q^x \cdot \text{Sym} \left[ \frac{\prod_{j=1}^k z_j^{\lfloor \frac{jd}{k} \rfloor - \lfloor \frac{(j-1)d}{k} \rfloor} \cdot z_{(n-x)a+1} \dots z_{(n-1)a+1}}{z_1^{-1} \left( \frac{z_1}{z_2} - q \right) \dots \left( \frac{z_{k-1}}{z_k} - q \right) z_k \cdot z_{(n-x)a} \dots z_{(n-1)a}} \prod_{i < j} \omega \left( \frac{z_i}{z_j} \right) \right]$$

$$P_{-k,d} = c_{k,d} \sum_{x=0}^{n-1} q^{n-x-1} \cdot \text{Sym} \left[ \frac{\prod_{j=1}^k z_{-j}^{\lfloor \frac{jd}{k} \rfloor - \lfloor \frac{(j-1)d}{k} \rfloor} \cdot z_{-(a+1)} \dots z_{-(xa+1)}}{\left( \frac{z_{-2}}{z_{-1}} - q \right) \dots \left( \frac{z_{-k}}{z_{-(k-1)}} - q \right) \cdot z_{-a} \dots z_{-xa}} \prod_{i < j} \omega \left( \frac{z_{-j}}{z_{-i}} \right) \right]$$

where:

$$c_{k,d} = \frac{(1 - q^{-1})(q_1 - 1)(q_2 - 1)}{(1 - q^{-1})^k (q_1^n - 1)(q_2^n - 1)}$$

Moreover, when  $k = 0$  we have  $P_{0,\pm d} = h_d^\pm$  (for  $d > 0$ ). Formulas (3.9) and (3.10) tell us how the above shuffle elements act on  $K$  in terms of tautological classes. When  $K$  is identified with the Fock space, it is more convenient to express this action in terms of the classes  $[\lambda]$  of torus fixed points, which are identified with MacDonal polynomials<sup>6</sup>. The easy case is  $k = 0$ , since  $h_d^\pm$  acts on  $\mathcal{M}$  by scalar multiplication, and therefore:

$$\left( \sum_{d \geq 0} z^{\mp d} P_{0,\pm d} \right) \cdot [\lambda] = [\lambda] \frac{\Lambda(\mathcal{W}|\lambda, zq)}{\Lambda(\mathcal{W}|\lambda, z)} = [\lambda] \prod_{i=1}^r \frac{1 - qzx_i}{1 - zx_i} \prod_{\square \in \lambda} \Omega \left( \frac{z}{\chi(\square)} \right)$$

**8.5.** We are therefore left with the case  $k \neq 0$ . Before we can state the general result, let us introduce some more notation. Given two partitions  $\lambda \leq \mu$ , we will write  $\mu - \lambda$  for the difference of the two partitions, perceived as a set of  $k$  boxes in the lattice plane. A standard Young tableau (denoted by SYT, plural SYTx) of shape  $\mu - \lambda$  is an arrangement of the numbers  $1, 2, \dots, k$  in the boxes of  $\mu - \lambda$ , in such a way that the numbers decrease as we go up or to the right. There is a bijection between SYTx and all collections of intermediary partitions:

$$\lambda = \rho_0 \leq \rho_1 \leq \dots \leq \rho_{k-1} \leq \rho_k = \mu$$

such that each partition has one more box than the one to the left. Naturally, this notion can be generalized to the  $r$ -tuples of partitions which index fixed points of  $\mathcal{M}$  (see Subsection 5.1). Given a SYT, we will denote by  $\chi_i$  the content of the box numbered by  $i$ . Then our general result is:

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<sup>6</sup>Here we use Haiman's renormalization of MacDonal polynomials

**Proposition 8.6.** *For any Laurent polynomial  $m(z_1, \dots, z_k)$ , let us consider the shuffle elements <sup>7</sup>:*

$$P^+ = \text{Sym} \left[ \frac{m(z_1, \dots, z_k)}{\left(\frac{z_1}{z_2} - q\right) \dots \left(\frac{z_{k-1}}{z_k} - q\right)} \prod_{i < j} \omega \left( \frac{z_i}{z_j} \right) \right] \in \mathcal{A}_k^+$$

$$P^- = \text{Sym} \left[ \frac{m(z_{-1}, \dots, z_{-k})}{\left(\frac{z_{-2}}{z_{-1}} - q\right) \dots \left(\frac{z_{-k}}{z_{-(k-1)}} - q\right)} \prod_{i < j} \omega \left( \frac{z_{-j}}{z_{-i}} \right) \right] \in \mathcal{A}_k^-$$

Then for any partitions  $\lambda$  and  $\mu$ , the matrix coefficient of the basis vector  $[\mu]$  in the  $K$ -theory class  $P^\pm([\lambda])$  equals:

$$\langle \mu | P^+ | \lambda \rangle = (q-1)^k \sum_{\text{SYT of shape } \mu - \lambda} \frac{m(\chi_1, \dots, \chi_k)}{\left(\frac{\chi_1}{\chi_2} - q\right) \dots \left(\frac{\chi_{k-1}}{\chi_k} - q\right)}$$

$$\prod_{1 \leq i < j \leq k} \omega \left( \frac{\chi_i}{\chi_j} \right) \cdot \prod_{i=1}^k \frac{\prod_{\square \text{ outer corner of } \lambda} (1 - \chi_i \chi(\square)^{-1})}{\prod_{\square \text{ inner corner of } \lambda} (1 - \chi_i \chi(\square)^{-1})}$$

$$\langle \mu | P^- | \lambda \rangle = (1 - q^{-1})^k \sum_{\text{SYT of shape } \lambda - \mu} \frac{m(\chi_1, \dots, \chi_k)}{\left(\frac{\chi_2}{\chi_1} - q\right) \dots \left(\frac{\chi_k}{\chi_{k-1}} - q\right)}$$

$$\prod_{1 \leq i < j \leq k} \omega \left( \frac{\chi_j}{\chi_i} \right) \cdot \prod_{i=1}^k \frac{\prod_{\square \text{ inner corner of } \lambda} (1 - q^{-1} \chi_i^{-1} \chi(\square))}{\prod_{\square \text{ outer corner of } \lambda} (1 - q^{-1} \chi_i^{-1} \chi(\square))}$$

In particular, the matrix coefficient vanishes unless  $\lambda \leq \mu$  (when the sign is  $+$ ) or  $\mu \leq \lambda$  (when the sign is  $-$ ). Let us note that among the factors of each summand in the above there are certain zeroes (there are  $k$  more zeroes in the denominator than in the numerator), which have to be removed from the above formula in order to make sense.

**Proof** According to Conjecture 2.6 of [6], any  $P^\pm$  as above can be written as a sum of products of shuffle elements of degree 1 <sup>8</sup>. Then we can prove Proposition 8.6 by induction on  $k$ . The induction step is straightforward, so we will simply do the case  $k = 1$ .

By Theorem 6.10, the operator  $P^\pm$  acts by a constant multiple of the simple correspondence  $\mathfrak{Z}^1 = \{(\mathcal{F} \supset \mathcal{F}')\}$ . Equivariant localization tells us that:

$$P^\pm([\lambda]) \sim \sum_{\mu} [\mu] \cdot \frac{\Lambda(T_\lambda \mathcal{M}, 1)}{\Lambda(T_{\lambda, \mu} \mathfrak{Z}^1, 1)}$$

The ratio on the right can be computed using (6.8), and we obtain:

<sup>7</sup>This is actually no restriction. As suggested by Conjecture 2.6 of [6], all shuffle elements can be written in this form. It is just that this form makes them particularly suitable for our computation, and it is the most relevant form in our quest for the  $P_{k,d}$ 's

<sup>8</sup>Even if we do not accept the Conjecture, the elements  $P_{k,d}$  we are interested in are of this form, as shown in the main theorem of [6]

$$P^\pm([\lambda]) = \sum_{\mu} [\mu] \cdot (1 - 1) \cdot \Lambda(\mp \mathcal{W}_{\lambda}^{\pm}, q^{\varepsilon-1} l^{\pm})$$

which is precisely the desired formula. The factor  $1 - 1 = 0$  is there in order to kill the zero in the denominator of  $\lambda(\mp \mathcal{W}_{\lambda}^{\pm})$ . This is the sense in which we have to remove zeroes from the formulas of Proposition 8.6.

□

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